Real Analysis II
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Lecture - 23.1

Curves

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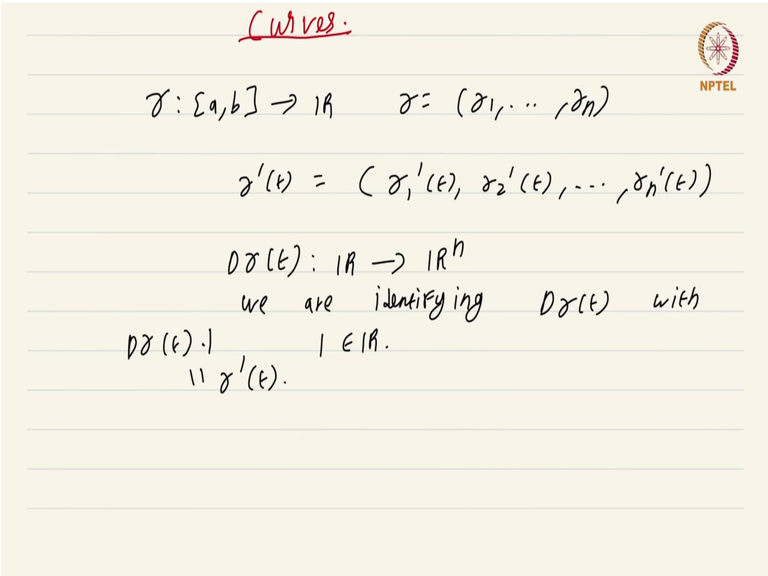

In this video, we shall discuss curves and in this series of videos about curves, we shall end with how to differentiate sorry how to integrate a differential form on a curve. So, let us recall some notation so that things become unambiguous and clear. So, if you have gamma from a b to R, we will use gamma prime of t to mean just gamma 1 prime of t, gamma 2 prime of t dot dot dot gamma n prime of t.

Just differentiate each coordinate ok. Here of course, gamma is gamma 1 dot dot dot gamma n. I am assuming of course, that each one of these functions gamma 1 to gamma n are differentiable on the close interval a, b. Of course, you can also view this gamma prime of t as d gamma at t; d gamma of t if you recall will be a linear map from R to R n ok.

So, essentially, what is happening is we are identifying, we are identifying, we are identifying d gamma of t with d gamma of t acting on the vector 1. The vector 1 is an element of R, it is also basis for R ok and if you just unwind all the definitions, it will be very clear to you that when d gamma of t acts on 1, what you get is nothing but what we have called gamma prime of t ok.

So, this is just some notation. You can view this derivative gamma prime of t as the tangent vector to the curve gamma at the point t. So, this viewpoint of viewing the derivative as both a vector and a linear map will become very important soon ok. Also, because many of this is many of these notations are motivated from physics, this vector gamma prime of t is also called the velocity of the curve at the point t.

So, it is good to use such terminology because it first of all it livens up and otherwise, boring discussion on differential forms. Moreover, it suggests you intuitive pictures that could prove very useful as a guide to correct proofs ok.

With that being said, this is the central topic of this lecture curves. So, let me define precisely, the various types of curves that we will be considering in this set of videos. So, this is going to be an entire definition of all the type of curves that we are going to see.

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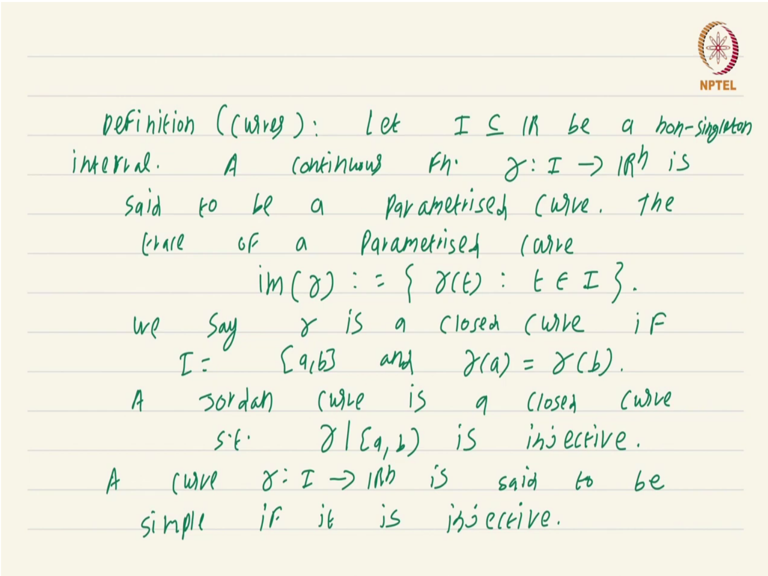

So, definition this is the definition of curves. Let I subset of R be a non-singleton interval. It could be anything; it could be bounded, unbounded; closed, half closed, half open, I do not care. A continuous function gamma from I to R n is said to be a parametrized curve.

You will notice this additional word parametrized in front of the word curve; you will understand in a moment, why we are distinguishing curves with this special word parameterized. The trace of a curve the trace of a parametrized curve this is usually denoted image of gamma. This is nothing but by definition, the set of points gamma of t as t runs through I.

So, in essence, the curve is the function gamma from I to R n, but the term curve conjures up in our mind and image of a squiggly sort of thing floating in space. That squiggly thing floating in space is this image of gamma.

For the purposes of these set of videos, it is important to distinguish between a parameterized curve and its trace. Often, when we speak of the trace, we will call it a curve; but whenever we are in proofs, a curve henceforth, always means a function gamma from I to R n ok. So, this is crucial.

We say gamma is a closed parametrized curve. Let me just shorten, since I have said that hence fourth curve should always be parametrized curve, let me just shorten it to closed curve. We say gamma is a closed curve, if I is some closed interval a, b and gamma of a equals gamma of b. Utterly, straightforward definition.

The starting point and ending point are the same. A Jordan curve; a Jordan curve is a closed curve; is a closed curve such that gamma of a, b is injected; gamma restricted sorry gamma restricted to closed a open b is injective, then we say that the curve is a Jordan curve.

So, a Jordan curve is a closed curve with no self intersections and the starting point and the ending point are the same. A curve is said to be simple, a curve gamma from I to R n is said to be simple is said to be simple, if it is injective.

So, simple curve is one which has no self intersection and of course, the starting point and the ending point, if there are any starting point and ending points because I could very well be a non-closed interval, the starting point and the ending point cannot be same.

Nevertheless, I caution the listener that many text books do call Jordan curve also has a simple closed curve. So, that differs from our terminology of what simple means. In our terminology, Jordan curve is not a simple closed curve; but unwilling to break tradition, we may also sometimes call a Jordan curve a simple closed curve. This should not cause no confusion ok. Now, these are just continuous functions so far ok; no differentiability conditions have been imposed. Now, we will impose differentiability conditions.

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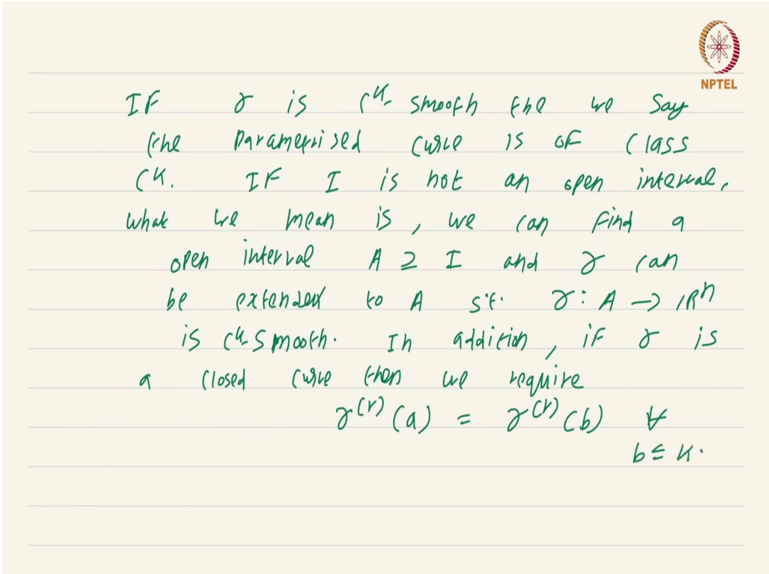

If gamma is C k smooth, then we say the parametrized curve is of is this parametrized curve is of class C k; class C k ok. Now, we need to understand what this means. Now, first let me clarify one more thing. If I is not, if I is not an open interval; I is not an open interval ok, what we mean is; what we mean is we can find; we can find a open interval A that contains I and gamma can be extended to A such that gamma, now the extended gamma from A to R n is C k smooth.

Now, what does C k smooth in this context mean? Well, it just means that gamma 1, gamma 2 dot dot dot gamma n are C k smooth ok. In addition if gamma is a closed curve; is a closed curve, then we require gamma r, the rth derivative of gamma at the point a to be equal to the rth derivative of gamma at the point b, for all b less than or equal to k. So, what this is saying is best illustrated by just a simple picture rather than me saying it in words.

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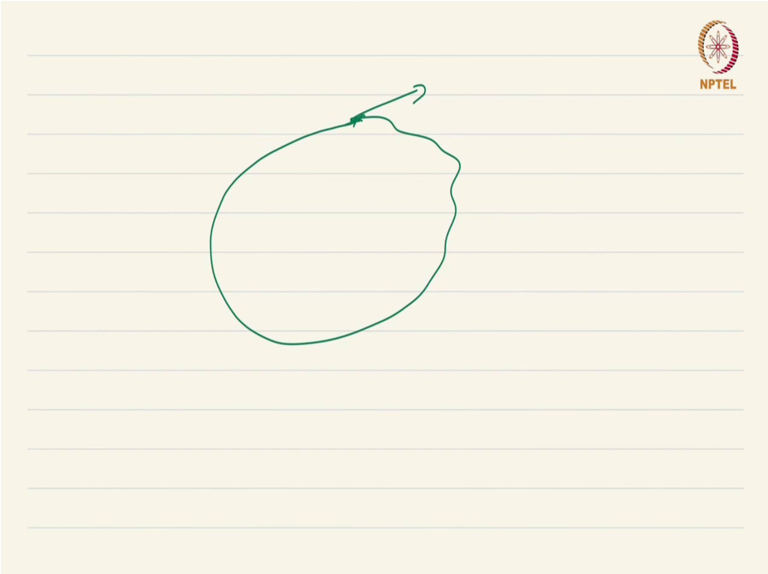

So, suppose this is your curve, what we require when you have a C k smooth closed curve is that at the ending points, the tangent vector should be the same, even if you take the derivative at a or the derivative at b. Not only should the tangent derivative, the tangent vectors be exactly the same, the second derivative should also be exactly the same, the third derivative should also be exactly the same.

So, you essentially want the derivatives up to order k at the end points to coincide and of course, you can make sense of these higher derivatives. You just differentiate coordinate wise because gamma is just gamma one to gamma n ok.

So, once again I caution the reader, a parametrized curve and its trace are different things. A parametrized curve is a function; the trace is a set ok. But nevertheless, many times when we are speaking about curves, we might mean the trace. So, the reader should always be vigilant to understand what I mean by the term curve.

But in general, when I say curve, I will always mean parametrized curve unless otherwise. So, this is quite confusing; but this is just a warning to be prepared ok. Since, I keep saying that a parametrized curve could be different from its trace, a good way to see this is to see through several examples.

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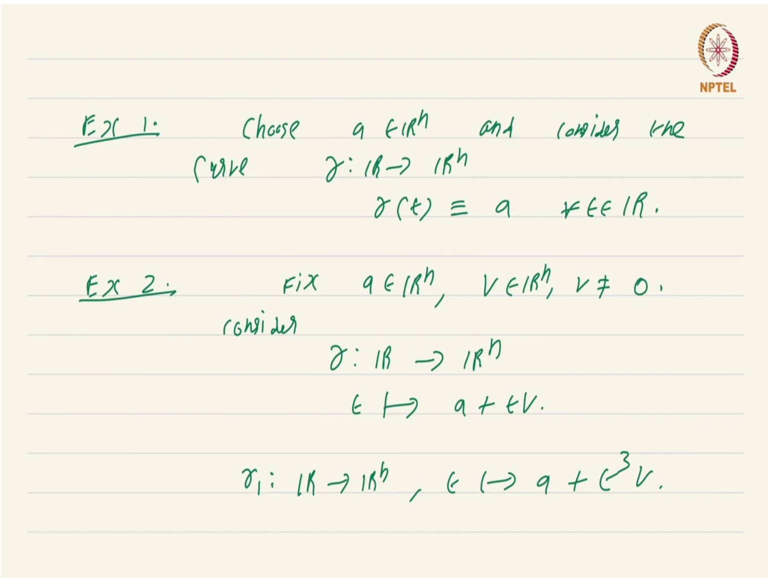

So, let us see some examples. Example 1, the simplest curve is you just pick if a set, I mean choose a in; choose a in R n choose a in R n and consider the curve gamma from R to R n which does nothing; gamma of t is just a for all t in R. This is just a point. This is not really a curve; this is a trivial curve. But of course, the first example should always be a trivial example. This is a golden rule of mathematics; the first commandment of mathematics.

Example 2, so let me just put the straight line that is the next simplest example. So, fix a in R n comma v in R n, v not equal to 0 ok and consider gamma from R to R n given by t goes to a plus t v. As you can visualize this just traces out the line passing through the point a in the direction v because I want this example to be genuinely different from the first example, I have insisted that v is not 0 that is the reason.

Now, needless to say in the first example as well as the second example, these are simple; I mean the first example is not a simple curve, the second example is going to be a simple curve and both example 1 and example 2, they are actually C infinity smooth curves ok.

Now, we can obtain the same trace as the straight line passing through a in the direction v by considering the new gamma 1 from R to R n given by t going to a plus t cube v as you can easily check this has the exact same trace, both traces coincide with the straight line passing through the point a in the direction v. Not only that, both curves are simple and C infinity smooth. So, here is a concrete example of curves that have the same trace, but are nevertheless quite different ok.

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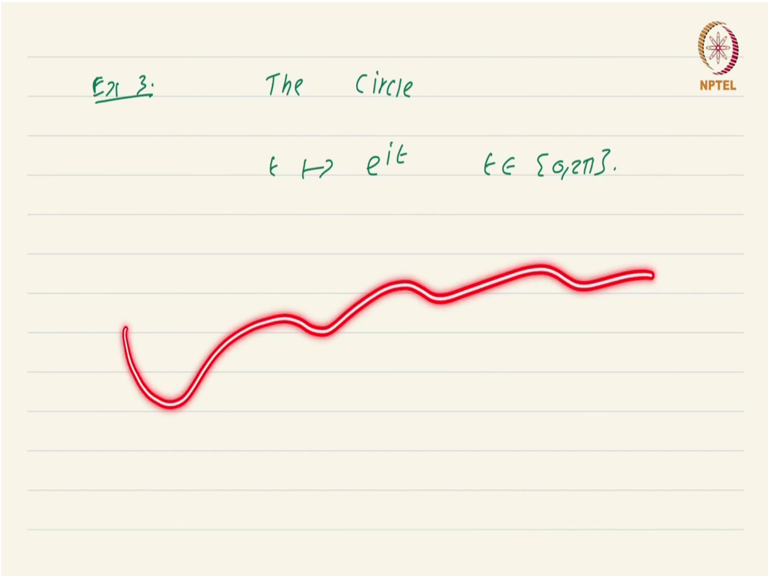

Now, the simplest and the most famous example, in fact, the most famous example in all of mathematics is what is coming up its pretty much unavoidable. This object is pretty much unavoidable irrespective of what branch of mathematics you study.

This is the circle. See here, I am conflating the image, the trace with the curve because I am saying the circle, but again such usage is common and you should be prepared to untangle the meaning each time such ambiguous notation is used.

So, the parametrized circle, however, is given by this curve t going to e power i t; t running from 0 to 2 pi. This is a Jordan curve and this curve is C infinity smooth, both of which you can easily check from the basic properties of cosine and sine. Of course, this curve t going to e power i t traverses the circle in a clockwise direction.

Now, as I have mentioned before, we would prefer to stick to visual terminology from physics calling tangent vectors as velocity vectors. For that same reason, this parameter which I am calling it t, it is for time. So, one good thing is to visualize these curves as a particle moving in space and leaving behind the trajectory. So, imagine a jet which is leaving out exhaust fume.

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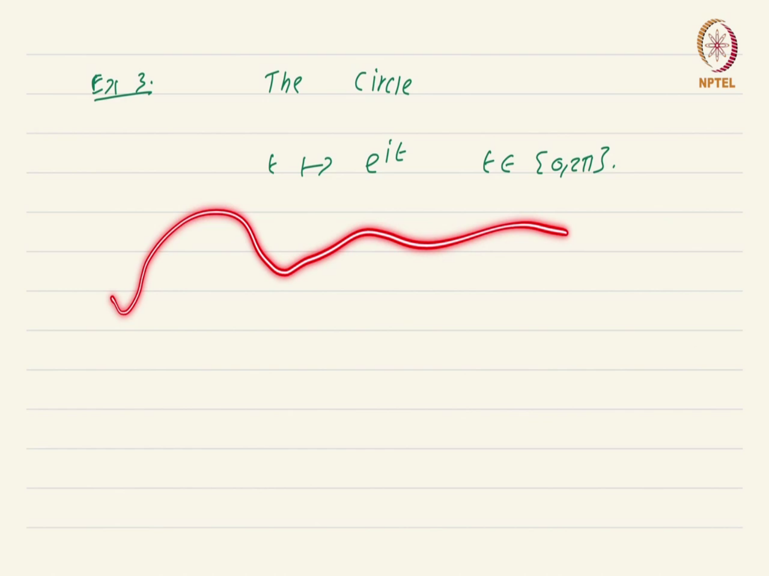

So, that is a good way to visualize curves.

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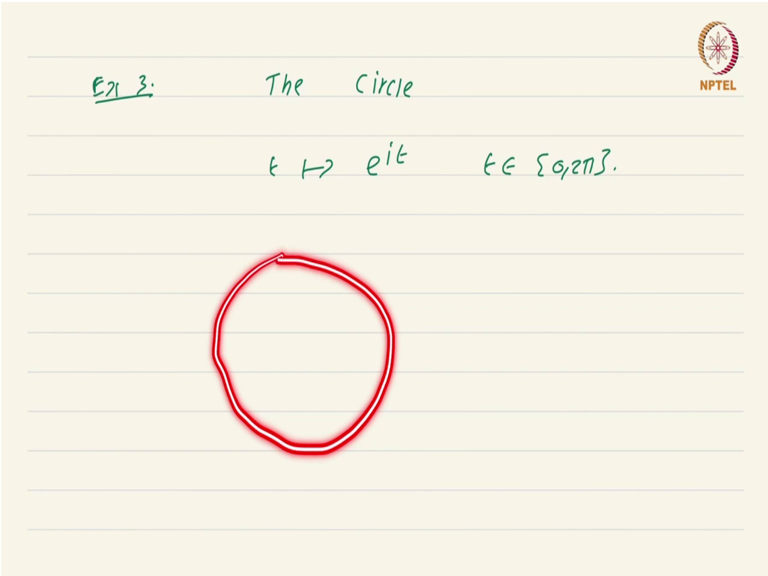

So, this t going to e power i t traces out the circle in a anticlockwise direction once ok.

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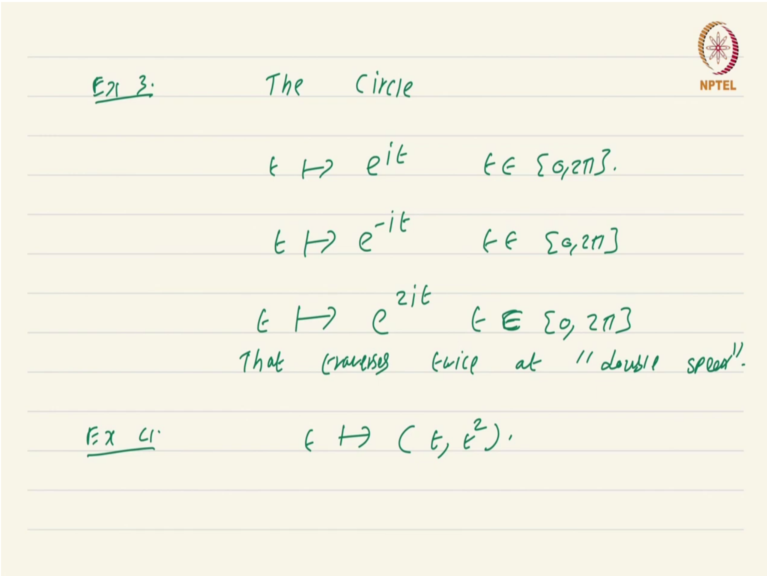

Of course, you can trace out the circle in a clockwise direction also by considering t going to e power minus i t; t coming from 0 to 2 pi. This is another Jordan curve which is C infinity smooth and has the exact same trace as the circle. But these two curves are quite different, they represent motion in two different directions ok. So, this is a good to keep in mind.

Now, you can traverse the circle in such a way that it is no longer a Jordan curve by doing t going to e power 2 i t. This is a parametrization of the circle that traverses twice, that traverses twice at double speed, at double speed ok. So, at this point, I must make some more comments because we are using terminology that could be interpreted in different ways.

When a topologist says Jordan curve, they mean the image. So, for a topologist, the circle will always be a Jordan curve. The definition, in fact, of a Jordan curve for a topologist is a homeomorphic image of the unit circle.

So, for a topologist, circle will always mean a Jordan curve, I mean circle will always be a Jordan curve; whereas, for us, because we are choosing to treat the curve via parametrizations, there will be a slight conflict of terminology and do not worry if all this sounds a bit confusing, much of this terminology in this course is designed to make what follows easy.

So, for the purposes of this course, if you just stick to the terminology I am giving, things will be a bit smooth ok. Now, let us see another famous example. This is the example of the upward facing parabola. So, this is the curve t going to t comma t square ok. So, this is the upward facing parabola as you can see. In fact, this example motivates the next class of examples.

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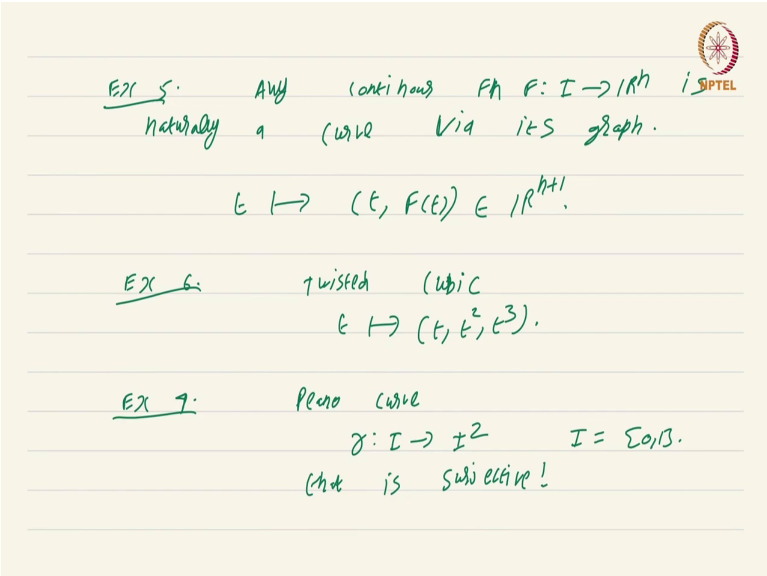

So, I would always like to give infinitely many examples. Example 5, any continuous function F from the interval I to R n is naturally a curve via its graph; via its graph that is look at the map t going to t comma f of t. This is a curve in R n plus 1. So, whenever you have a continuous function from an interval to R n, you can view its graph in a natural way as a curve ok and this will be a simple curve as well. This will be a simple curve.

So, this is a good thing to know. Another example, this is a very famous curve called the twisted cubic. I recommend you use some program like wolfram alpha to draw this and see how it looks this is the twisted cubic, this is the curve t going to t, t square, t cube ok. So, there are many many many examples of curves which have been studied for over 2000 years. So, it would not I, since I do not have 2000 years to finish this course, I cannot list all of them.

So, I strongly recommend that you pick up some book on differential geometry, basic differential geometry; the books by Abati, do Carmo or Barrett O'Neill, they are excellent books and you will find plenty of classical curves and where they arise in physical phenomena as well. So, please do that is to have a more bigger repertoire of curves in your pocket. Final example is an investigatory project for you. All these examples might tempt you into thinking that a parametrized curve is a nice one-dimensional object in space. That is not so.

There is a famous curve called the peano curve, this is a curve gamma from I to I 2; where, I is just close 0, 1 ok that is surjective. Wow. So, there is a curve from the close interval 0, 1 to the square 0 1 cross 0 1 that is surjective. So, this is an extremely weird curve. In fact, the creation of this curve shocked many mathematicians of that era.

So, these curves are called space filling curves and are interesting to study from the perspective of topology. I will not pursue this any further, leaving it up to you to figure out how this curve is constructed. Now, one final remark you might think that ok, this curve gamma from I to I 2 is very complicated; but it is certainly not going to be injective, so that is fine right.

So, if you just stick to simple curves, then they will be within quote simple. Not true, you could have a simple curve from I to R n which is not one-dimensional. In the sense that the Lebesgue measure of the image is positive that is the Lebesgue dimension is positive. I am not going to talk about this any further; you could not you could not you could have a curve in R 2, whose two-dimensional Lebesgue measure is actually non-zero.

This cannot happen for C 1 smooth curves that you can prove; but for just a continuous curve, even the additional hypothesis of injectivity does not help bring down the dimension in some sense ok. So, these are the definition, the basic definition of the curves, we will be considering in the rest of this course on vector analysis and some examples to keep in mind, when you study the abstract theory.

This is a course on Real Analysis and you have just watched the video on curves.