Real Analysis II
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Lecture – 2.2
Examples of Normed Vector Spaces

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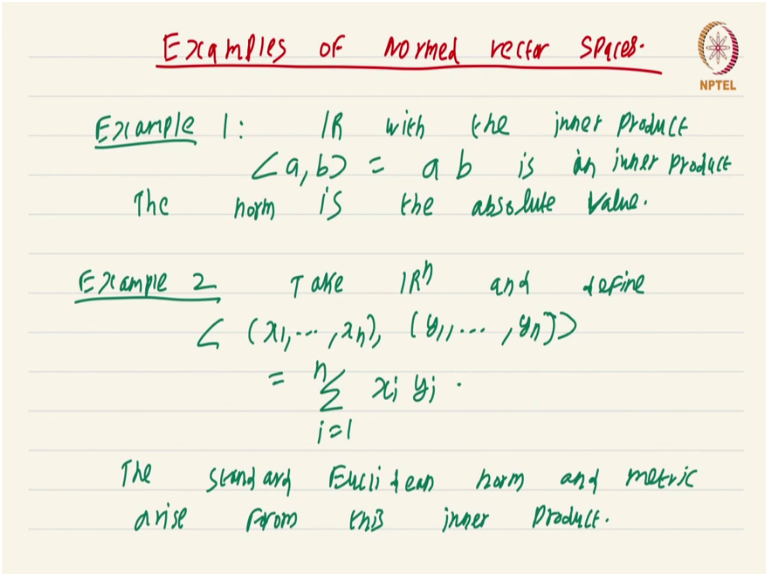

In this video, we shall see several examples of normed vector spaces. Let us begin by taking a very basic example.

Example 1: $\mathbb{R}$ with the inner product

$$\langle a, b \rangle = ab$$

is an inner product and the associated norm is the absolute. This is the simplest and most basic example of an inner product space and the associated norm.

Example 2: It is something that you have explored in great detail, probably in a vector calculus course or multivariable calculus course in your more basic courses. Take $\mathbb{R}^n$ and define

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{i=1}^{n} x_i y_i.$$

This is the standard inner product on $\mathbb{R}^n$. The fact that this is an inner product is rather easy to see, and you can see that the Euclidean norm and metric arise from this inner product. These are all easy checks that I am leaving to you.

So, these are the standard prototype examples of inner product spaces and norms. These are the prototypes on which the more abstract definition that we studied is based.

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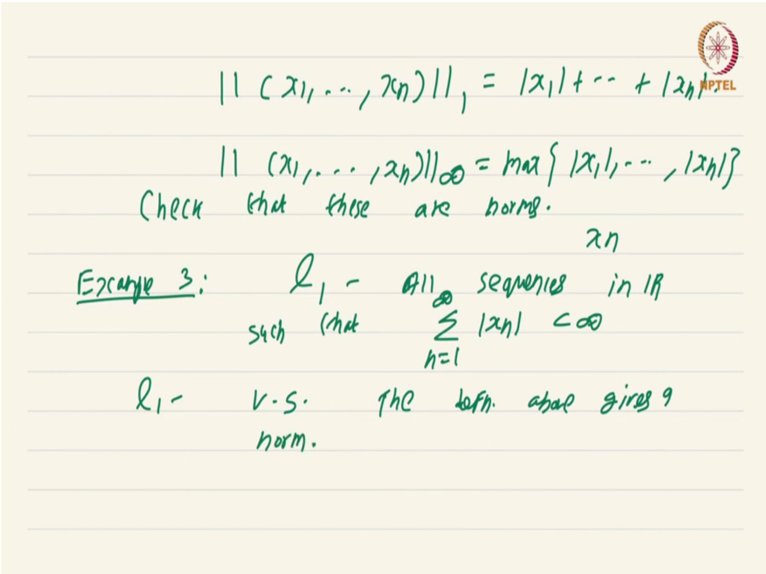

So, let me just remark that there are several other natural norms on $\mathbb{R}^n$. Let me just list two of them. And we will sort of explore this in some detail when we come to product metric spaces. You can define the l_1 norm. This is nothing but the sum of the absolute values of the various components.

$$\|(x_1, \dots, x_n)\|_1 = |x_1| + \dots + |x_n|.$$

The other one is called the l infinity norm on the Euclidean space, denoted like l_∞. This with an infinity this is nothing but the maximum of the various components

$$\|(x_1, \dots, x_n)\|_\infty = \max\{|x_1|, \dots, |x_n|\}.$$

So, check that these are norms. Now both of these norms do not arise from an inner product. I will maybe put this as an exercise for you. These both do not arise from an inner product; nevertheless, they are natural and interesting norms, and in some context, using these norms will make our lives easier.

So, we will use these as and when it is appropriate. From the perspective of metric spaces, all these norms are sort of going to be equivalent in a sense that we shall define a bit later. So, from our purposes of studying the topology of these spaces, they will all be identical. It does not matter which one of these norms, whether you put these l_1 norm or the l_∞ norm or whether you put the norm that we saw before the Euclidean norm, it does not matter. All of these are going to be equivalent.

So, let us move on to the next example,

Example 3: We denote by l_1, all sequences in $\mathbb{R}$ such that $\sum_{n=1}^{\infty}|x_n| < \infty$. In other words, this l_1 is the collection of all sequences whose corresponding series is absolutely convergent. Now it is an easy exercise for you to see that l_1 is a vector space with component-wise addition, and the definition that we gave will give you a norm. So, the definition above gives a norm. These are all easy checks that again, I am going to leave it to you.

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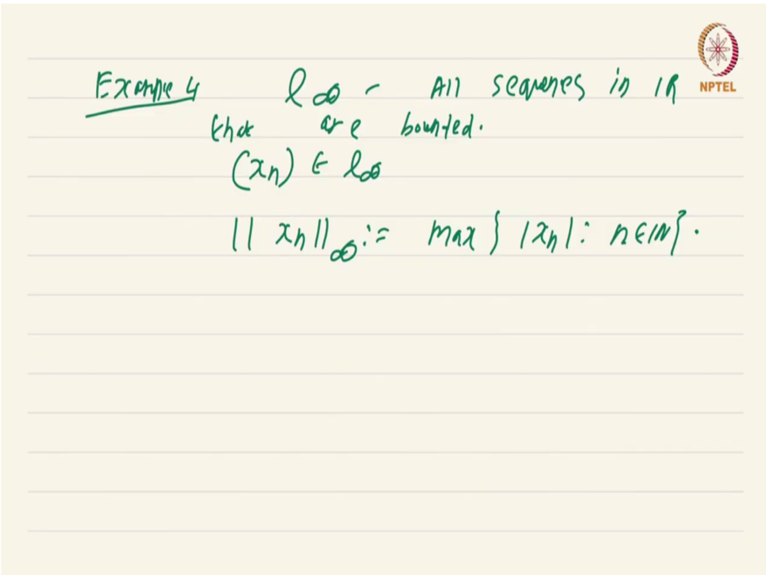

Example 4: This is called l_∞ and if you have been following the notations that I am giving, you can sort of guess what this is going to be. This is again going to be all sequences in $\mathbb{R}$ that are bounded. Again it is very easy to see that this will be a vector space with component-wise addition and component-wise scalar multiplication.

And what you do is if $(x_n) \in l_\infty$, we define

$$\|x_n\|_\infty = \max\{|x_n| : n \in \mathbb{N}\}.$$

You take the highest absolute value quantity in the sequence. So, since this is bounded, such a maximum will exist. To be 100 percent precise, I should not use maximum.

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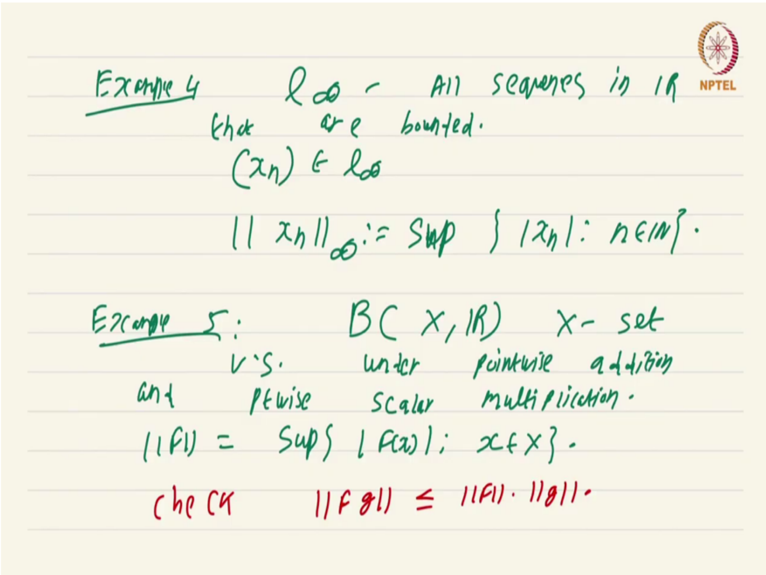

I should use supremum because it is quite conceivable that this highest value is not part of the sequence. For instance, if you can just take the sequence $\left(1 - \frac{1}{n}\right)$, the norm of that sequence is 1, and 1 is not an element of the sequence. So, I want you to check the details of these examples to make sure that you understand why, in each case, the space under consideration is a vector space and why, indeed that it is a norm.

Now the example $B(X, \mathbb{R})$ which we have already seen.

Example 5: $B(X, \mathbb{R})$, where X is a set, and $B(X, \mathbb{R})$ is the set of all bounded functions on X. This is going to be a vector space under point-wise addition, point-wise addition, and point-wise scalar multiplication. That is easy to see. If you recall, we had defined a sup norm on this space. Let me recall

$$\|f\| = sup\{\,|f(x)| : x \in X\}.$$

This was the norm of a bounded function defined on the set X. Now, you can check that, indeed, this norm satisfies all the properties of a norm, and therefore, you get a metric that we have discussed before. Now one interesting thing is on this space $B(X, \mathbb{R})$ you can multiply two functions. So,

Exercise: I want you to check that $\|fg\| \leq \|f\| \times \|g\|$ and that inequality need not always be an equality. Think of an explicit example where this is not going to be an equality.

Please do this as an interesting and basic exercise. So, again I want you to check example 5 also in detail. In some sense example, 5 is the most important example from our perspective. Now, let me make a general definition that will come again and again in this course and elsewhere in topology.

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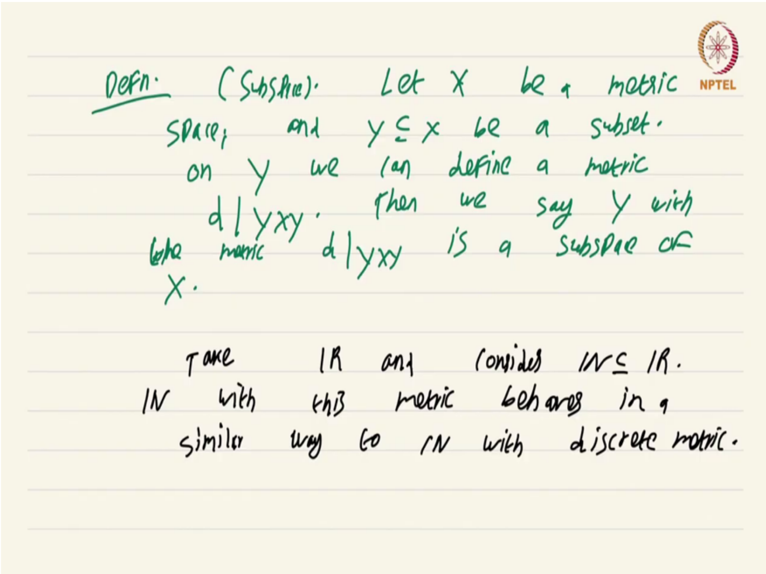

Definition: This is the definition of a subspace. Let X be a metric space and $Y \subseteq X$. On Y we can define a metric. This is just $d|_{Y \times Y}$. You have the function $d : X \times X \to \mathbb{R}$. We can restrict it to this product $Y \times Y$. . The fact that $d|_{Y \times Y}$ is going to be a metric is obvious. Then we say Y with the metric $d|_{Y \times Y}$ is a subspace of X.

So, this is just there was no need to, I mean, take out time and highlight this as a definition. Whenever you have a subset of a metric space, you can make it into a metric space naturally by taking the metric in the ambient space X and just restricting it to Y. Now, we will study all sorts of subspaces in this course, but there are many interesting subspaces of just $\mathbb{R}^n$.

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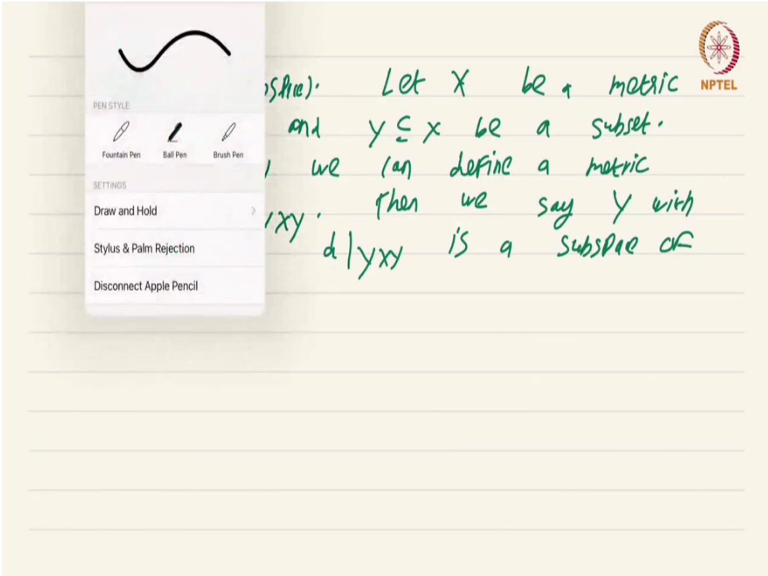

So, let me just highlight a few interesting subspaces of just $\mathbb{R}^n$. Take $\mathbb{R}^n$ or take just $\mathbb{R}$ and consider $\mathbb{N} \subset \mathbb{R}$, the natural numbers. Now, you can restrict the metric on $\mathbb{R}$ to these natural numbers, and you will notice that when you restrict it, this space $\mathbb{N}$ behaves very similarly. So, $\mathbb{N}$ with this metric behaves similarly to $\mathbb{N}$ with discrete metric.

Now, I am not going to make this precise. What I am going to ask you to do is think about this and make this precise. What is the meaning of N with this metric that looks very similar to the discrete metric? Can you make a more precise statement? I am going to leave it to you as an exercise.

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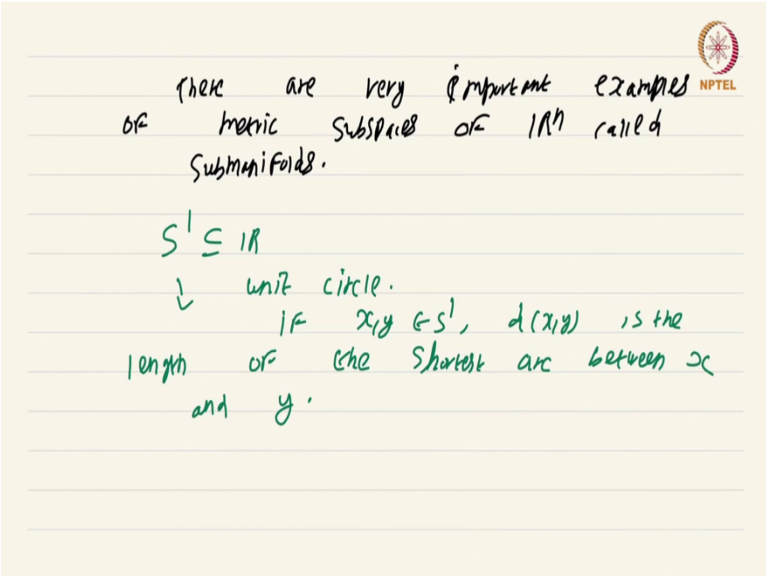

Now coming back to the general scenario of $\mathbb{R}^n$. There are certain very important examples of metric subspaces of $\mathbb{R}^n$ called submanifolds. So, rather than spending an elongated amount of time defining what a submanifold is, I will say they are things like if you are taking $\mathbb{R}^2$ then nothing but things like circles and parabolas and ellipses.

These are things that are smooth curves that are sitting inside $\mathbb{R}^2$. Similarly, in a higher dimension, all these nice surfaces like spheres and cylinders and the higher dimensional analogs of these are all subspaces called submanifolds locally. They will look like Euclidean space.

I am not going to make this precise in this course. On these, you always have a metric coming from the subspace coming from the ambient metric on $\mathbb{R}^n$ you can put a metric on the sub-manifolds. Now, these submanifolds are very important, and they are studied in differential topology and differential geometry.

Now, often on these subspaces, the Euclidean metric is not the nicest metric to put on it. I will just give one example. Look at $S^1 \subseteq \mathbb{R}$, the unit circle. Now, we can define another metric on S^1 as follows. If $x, y \in S^1$ then $d(x, y)$ is the length of the shortest arc between x and y. There are two arcs on the circle that join x and y. Look at the length of the shorter one. You can spend some time and show that this definition will give you an alternative metric on the circle that is different from the one that comes from the ambient Euclidean space. Now, since we have dealt with subspaces, now when you make statements like let us say suppose I say I have this metric space X and I have a subspace $Y \subseteq X$.

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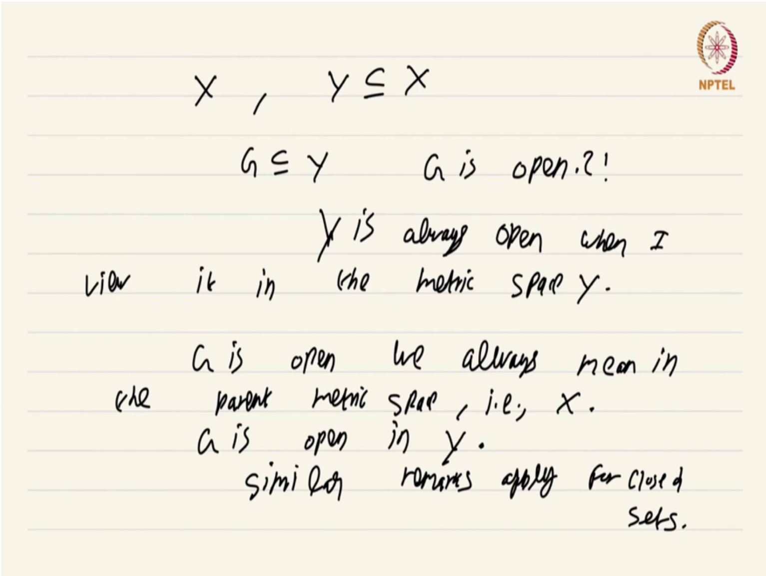

Now, if I have a set $G \subseteq Y$ and suppose I say G is open, what does this even mean? What does this even mean? This statement G is open could mean two things. Does it mean that G is open in Y, or does it mean that G is open in X? What I mean by that is, do I view G as a subset of Y with the subspace metric, or do I view G as a subspace of X; how do I view it? What is the statement open mean now?

To make this point that this is not a trivial statement. Observe that this Y if I treat it as a metric space in of itself, this Y is always. So, if this is somewhat tricky, it is not always true that if you have G which is open when you treat it as a subset of Y, it may not be open when you treat it as a subset of X. It depends on what is the parent metric space we are going to consider.

So, whenever you have this nested subspace as $Y \subseteq X$. So, you have potential ambiguity when we say G is open; we always mean in the parent metric space; that is X. So, whenever we just make a blanket statement, you look at the largest metric space we are considering, and it is open in that is what it means.

Now, if you want to say G considered as a subset of Y. You want to check whether it is an open set when you treat it as a subset of Y then we will say G is open in Y. So, if you want to distinguish that we are treating Y as the metric space under which we are analyzing G, we will be explicit, and we will say G is open in Y.

Similarly, the same remarks apply for closed sets, so similar remarks for closed sets. Do not make this mistake of thinking that open and close are absolute concepts. Open and closed always depend on which metric space we are considering. For that given situation, when there are multiple metric spaces, be very careful and think which metric space we are trying to make the statement under.

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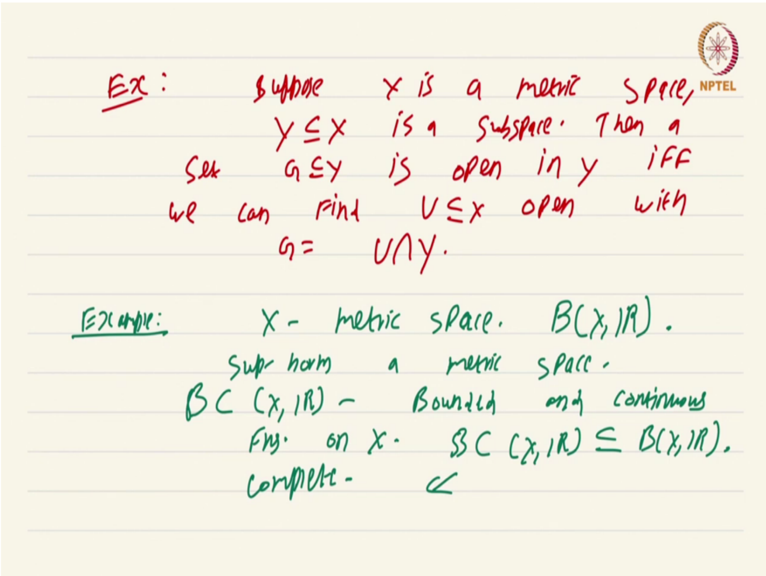

So, in this context, we have this nice exercise.

Exercise: Suppose X is a metric space, $Y \subseteq X$ is a subspace. So, we treat this subset as a metric space with the metric induced by the ambient metric space X. Then, a set $G \subseteq X$ is open in Y if and only if we can find $U \subseteq X$ open with $G = U \cap Y$.

So, intentionally I am not going to clarify the meanings of the occurrences of the word open. I want you to figure out which open each occurrence of the term open what it means. So, with that being said, let us move to the next example,

Example: We already considered the space $B(X, \mathbb{R})$, when X is any set. So, now, let X be a metric space. There is absolutely nothing stopping us from considering $B(X, \mathbb{R})$ again right. Just because X is a metric space does not mean that we cannot talk about $B(X, \mathbb{R})$. X just happens to have more structure than required. So, $B(X, \mathbb{R})$ with this sup, a metric space. Now, that is not the interesting bit. Consider $BC(X, \mathbb{R})$. It is bounded and continuous functions. That is one of the advantages of putting X as a metric space. We can talk about bounded and continuous functions on X. This $BC(X, \mathbb{R})$ is, of course, a subspace of $B(X, \mathbb{R})$. Not only is it going to be a subspace of $B(X, \mathbb{R})$, this is complete, which we will prove soon enough. This is again one of the most important examples of metric spaces and normed vector spaces.

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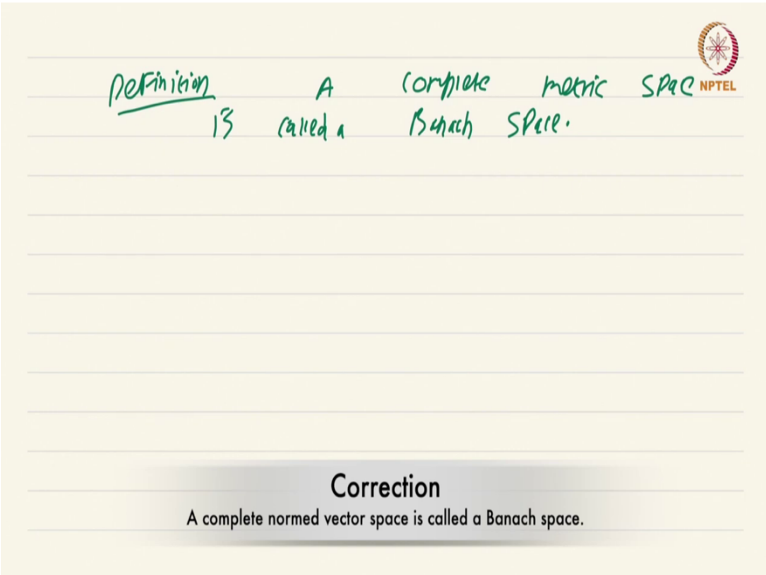

So, let me just make a definition in this context.

Definition: A complete normed vector space is called a Banach space.

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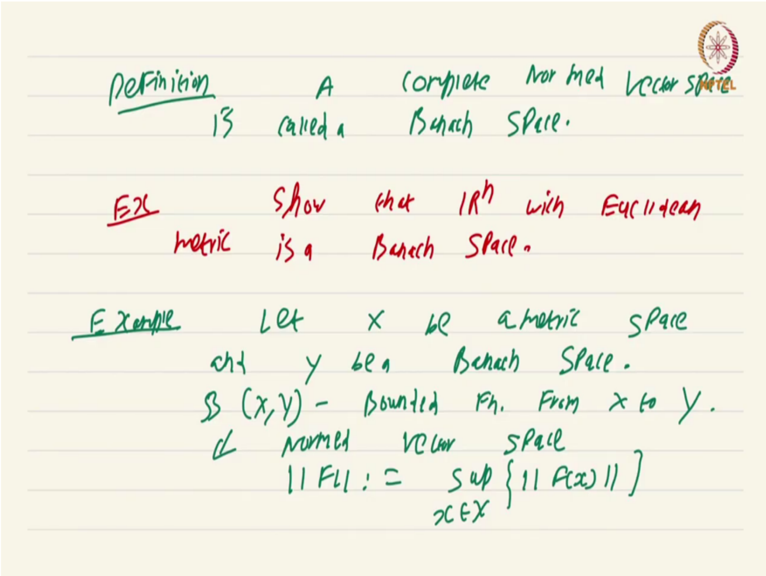

$\mathbb{R}$ is obviously, a Banach space.

Exercise: Show that $\mathbb{R}^n$ with Euclidean metric is a Banach space.

This is a rather important exercise but not very challenging. Now, we will generalize this last example a bit further, and this will be the final example of normed vector spaces that we are going to study.

Example: Let X be a metric space and Y be a Banach space. So, the co-domain is no longer real numbers, but it is a Banach space. Consider $B(X, Y)$ the collection of all bounded functions from X to Y. Because Y is a Banach space, there is a metric. You can talk about a bounded function. This just means that the range of a function is contained in some open ball centered at the origin of the space Y. That is what bounded function means.

So, you can check that this is a normed vector space. First, check that it is a vector space and it is also a normed vector space with the sup norm and the sup norm of a function f

$$\|f\| = sup\{ |f(x)|: x \in X\}.$$

You take all the possible values of f evaluated at various points of X. Because this is bounded, this entire quantity is going to be a finite number. You can check again that this is going to be a norm. These are all trivial checks; that is why I am going to leave it to you.

Note that because X is a metric space and Y is a Banach space; therefore, Y is also a metric space.

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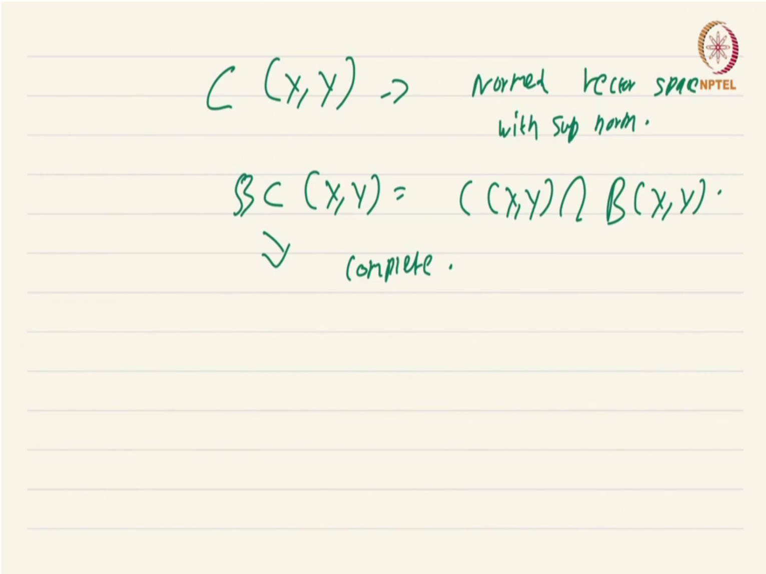

We can also talk about; we can also talk about continuous functions from X to Y. We can also talk about the space of continuous functions from X to Y, but because X is not is just a general metric space, we will later study compactness. If X is compact, then this also going to be a normed vector space with sup norm.

It is going to be a subspace of $B(X, Y)$ because continuous functions on compact sets will turn out to be bounded just as we have seen for the real numbers. However, if X is just a general metric space, there is no reason to believe that a continuous function will be bounded.

So, because of that, we have to consider $BC(X, Y)$. Again $BC(X, Y)$ is just $C(X, Y) \cap B(X, Y)$ and exactly as before, we see that this is also complete. Later towards the end of this set of videos on metric spaces, we will study the compact subsets of $BC(X, Y)$. That is the famous Ascoli-Arzela theorem.

So, with this collection of examples of normed vector spaces, I leave you with many minor details to check. I urge you to do it at least once in your life because these details might seem irrelevant to you, but trust me and take it from me that you must check it once. This is a course on Real analysis, and you have just watched the video on examples of Normed Vector Spaces.