Real Analysis II
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Lecture - 22.3
Completing the Squares

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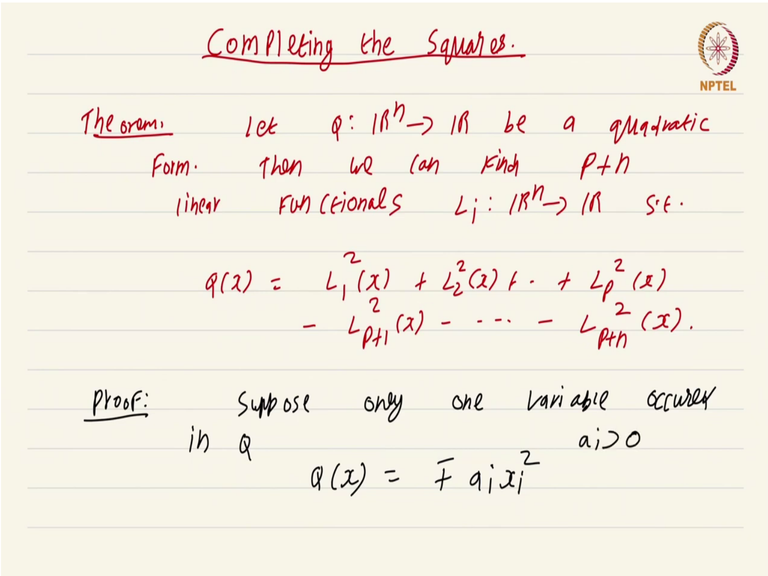

In this video, we are going to use a very old method called Completing the Squares to decompose a given quadratic form into a sum of squares of linear functionals. This method can also be used to show that any quadratic equation, the roots are given by the famous formula minus b plus or minus root of b squared minus 4ac by 2a. The same idea involved in the proof of this formula is what is involved here, except the situation is slightly more abstract.

So, let me state the theorem. So, the theorem is as follows, let Q from R n to R be a quadratic form; be a quadratic form. Suppose not suppose then we can find p plus n linear functionals. You will understand why I am writing it as p plus n in a moment, p plus n linear functionals L i from R n to R such that Q of x is nothing but L 1 x square plus L 2 square x plus dot dot dot L p squared x.

Afterwards the sum of squares starts to become actually a difference of squares. So, minus L p plus 1 squared x minus dot dot dot minus L p plus n squared x. So, there are p positive squares and p negative squares and all of them are sum together and that is going to be the quadratic form.

Let us go to the proof. The proof is as I said depends on a completion of squares. So, the formal proof proceeds by induction on the number of variables that occur in Q. So, suppose only one variable occurred in Q, in that case Q of x is has got to be plus or minus a i x i squared, where I runs from 1 to n, one of those numbers from 1 to n, it is got to be a square of those because its a quadratic form and there is a constant plus or minus a i, ok.

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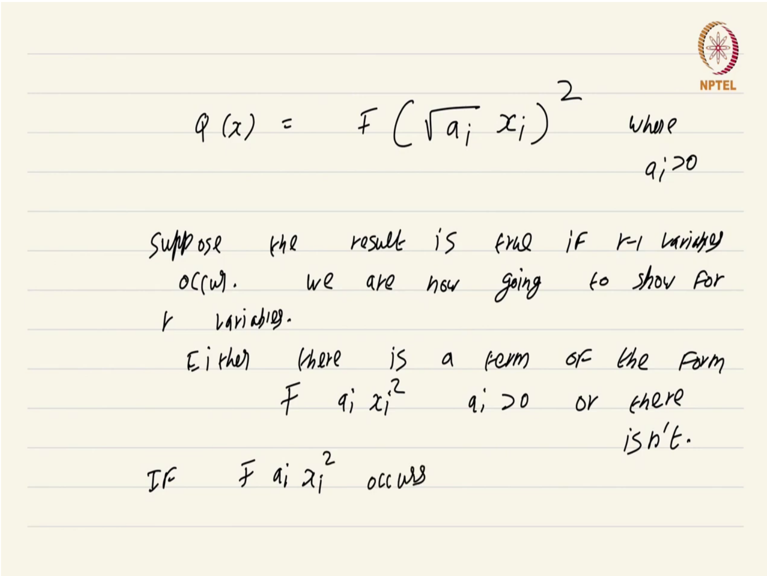

Now, the fact that this is a sum of squares is rather obvious, you can just write Q of x as you can write Q of x as plus or minus square root of a i x i squared, where a i is greater than 0, ok. So, I am I am assuming here also that a i is greater than 0, otherwise there was no reason to put a plus or minus in front if a i is allowed to be negative also, ok.

So, the case R equal to 1 is trivial. Now we are going to proceed by induction. So, suppose the result is true the result is true if R minus 1 variables occur. So, assume that the result will be true, if there are only R minus 1 variables occurring in the expression for Q. We are going to show it for R, ok. We are now going to show; going to show for R variables.

Now, there are only 2 possibilities, either there is a term; there is a term of the form; of the form plus or minus a i x i squared a i greater than 0 or there is not; there is not. These are the only 2 possibilities, either some term is a square or none of the terms are squares, and the every single term in this quadratic form is nothing but a mixed term of the type x i x j with some constant appearing in front.

Now, if plus or minus a i x i squared occurs, then we can sort of do a very simple completing the square just like what we did before. What we do is the following, ok.

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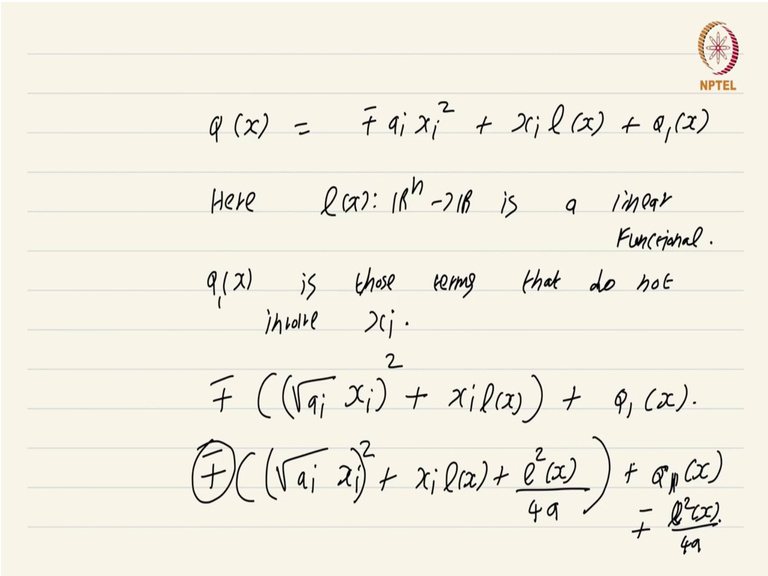

What we do is, we write Q of x as we write Q of x as this plus or minus a i x i squared plus x i l of x plus some Q 1 of x, ok. So, some explanation is in order. This l of x, so what I am doing to what I am going to do to obtain this l of x is, I am going to club together all the terms that involve x i but is not a square.

There is only one term that involves x i squared and that term I have already isolated it and put it here. I look at all the other terms where x i occurs and take x i outside, that can be done. So, what will be left is a linear functional.

So, here l of x from R n to R is a linear functional which could possibly be just identically 0. So, keep that in mind. Its not such an important point actually. And, Q 1 x; Q 1 x is those terms that do not involve x i; that do not involve x i, ok. So far so good. All I have done is I have club together all the terms that in which x i occur except the square term which I have put separately. Take an x i outside ended up with a linear functional and all the other terms I am putting together it as Q 1, ok.

Now, notice that by possibly replacing l of x by minus l of x, I can club together the first 2 terms also in the following way. I can write it as plus or minus square root of a i x i squared plus x i l of x plus Q 1 of x. So, to do this, I might have to put a minus sign in front of l of x to do this, but that is irrelevant, this can be done.

Once this is done, you complete the squares. What I want to do is, I want to write this term as a perfect square. So, I need to add and subtract something to make this into a perfect square and what I need to add and subtract is precisely l squared x by 4a. Think about why that is the case. Well, once you do that, you can write this as plus or minus square root of a i x i squared plus x i l of x plus l squared x by 4 a, ok. Here I forgot some brackets.

So, once you do this, of course, I have forgotten some terms, plus Q 1 of x; plus Q 1 of x and there is a plus or minus l squared x by 4a, now. That comes because I have done a plus or minus. So, technically to be 100 percent precise, this is a plus Q 1 x minus or plus l squared x by 4a, ok. Depending on what sign occurs here, the opposite sign will occur here, excellent. Now, we have I mean we are almost done, what we do is the following.

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$$\pm\left(\sqrt{a_i}\,x_i + \frac{l(x)}{2a}\right)^2 + Q_1(x) \mp \frac{l^2(x)}{4a}.$$

By induction we can write $Q_1(x)$ as a sum of squares and we are done.

Suppose no square term occurs. Suppose $\pm a x_i x_j$ is a term in Q, $i \neq j$, $a > 0$.

We complete the squares and write the first term as plus or minus square root of a i x i plus l of x by 2a the whole squared plus Q 1 of x minus or plus l squared x by 4 a, ok. So, what are the terms that remain? Well, it is just terms that involve no x i that is Q 1 x and l squared x by 4 a also does not involve any of the terms x i, right. Because we had already gotten rid of the x i by taking it out, whatever was left was this l of x.

So, by induction we can write; we can write Q 1 of x as a sum of squares; as a sum of squares and we are done; and we are done, ok. So, this concludes the proof under the scenario that there is a term of the form a i plus or minus a i x i square. Under that scenario, we have managed to write this quadratic form Q as a sum of squares of linear functionals, ok excellent.

Now, suppose no mixed term occurs, sorry not no mixed terms, suppose no square term occurs suppose no square term occurs, ok. So, this quadratic form it is a degree 2 thing, so, some term has to occur. So, suppose x i x j. So, some let us say plus minus a x i x j is a term in Q, ok. Here of course, i is not equal to j by assumption, no square term occurs. And, we are going to also assume as usual that a is greater than 0. Suppose we have such a square non square term.

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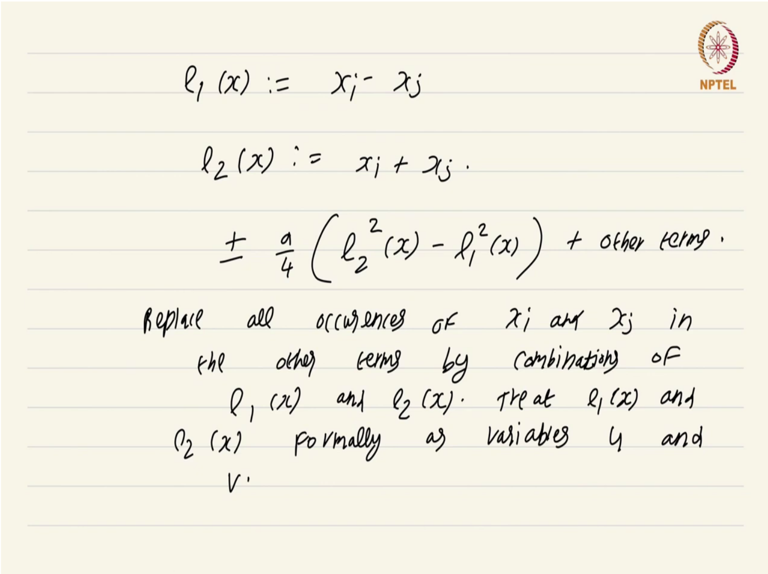

Now, what we are going to do is we are going to define some linear functionals, we are going to define l 1 of x to be nothing but x i minus x j and l 2 of x, this linear functional is nothing but x i plus x j, ok. Now, what you can do is, you can write down; you can write down this quadratic form as plus minus a by 4 into l 2 squared x minus l 1 squared x plus other terms, right. So, l 2 squared will be nothing but x i squared plus x j squared plus 2x i x j, right. And this would be x i squared plus x j squared minus 2x i x j.

So, when you subtract l 2 from l 1 sorry, when you subtract l 1 from l 2 those square terms will just cancel off, and you will be left with minus not minus plus 4 x i x j. So, that is why I had to put an a by 4 outside.

So, this is just elementary algebraic manipulation, ok. Now, what you do is replace all occurrences of x i and x j x i and x j in the other terms; in the other terms by combinations of l 1 x and l 2. For instance, x i is just l 1 plus l 2 by 2, right and x j you can think of a similar expression for x j.

Now, treat l 1 of x and l 2 of x formally as variables as variables u and v, ok. So, this part might sound a bit fishy or rather might stink a little bit, it might fee you might think that this is not rigorous. All I am doing is, I have these linear functionals l 1 and l 2. I am just calling them u and v formally.

So, if you observe that the proof in the previous part was completely formal, there was nothing about the meaning of the terms. It was just straightforward from algebraic manipulation. Now, what we do is we replace this l 1 and l 2 by this u and v.

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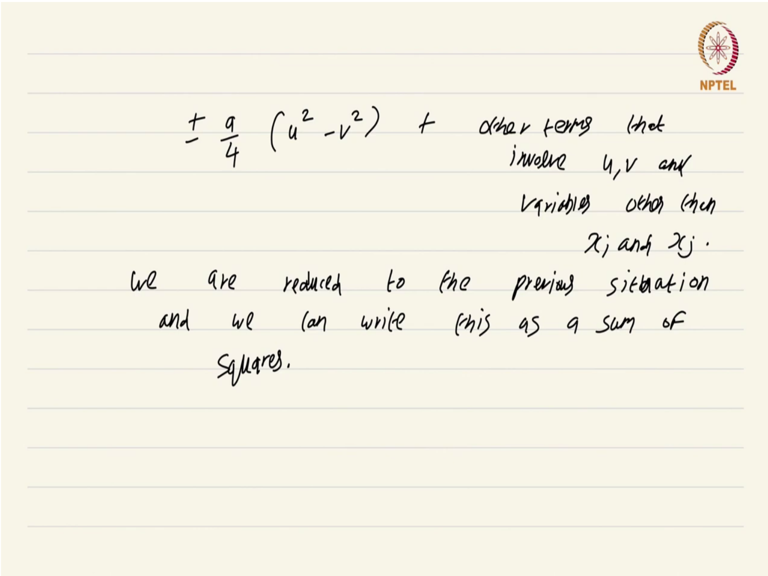

So, this quadratic form will just become plus or minus a by 4 into u squared minus v squared; u squared minus v squared plus other terms that involve u v and variables other than x i and x j, ok.

Now, we are reduced to the previous situation we are reduced reduce to the previous situation. There is a square term occurring. For instance, this u squared, and v squared both of them occur. We are reduced to the previous situation, and we can write this as a sum of squares; we can write this as a sum of squares.

And, once you write this as a sum of squares, all you do is replace back u and v by their expressions in terms of x i and x j, and you are done, right. You are absolutely done. You have formally manipulated this and obtained the quadratic form as a sum of squares. So, I want you to ponder about the last part of the proof, it is not that difficult, it is rather straightforward.

It is just you have to understand what this substitution really means, ok. So, tradition demands that I work out one or two examples. So, I will work out really trivial examples and leave some in the exercises to you.

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$$x^2 + xy$$

$$\left(x + \frac{y}{2}\right)^2 - (y/2)^2.$$

this quadratic form is indefinite. (why?).

$$x^2 + 4xy + 5y^2$$

$$(x+2y)^2 - 4y^2 + 5y^2 = \underbrace{(x+y)^2 + y^2}_{\text{positive definite}}$$

So, look at the quadratic form x squared plus xy, ok. Now to complete the squares its very easy, all you do is you write this as x plus y by 2 squared minus y by 2 squared, ok. So, this was a rather straightforward thing. This so, this quadratic form is indefinite, this quadratic form is indefinite, ok. So, why is it indefinite? Think about why this is true, think about why this is true.

So, another example. Look at the quadratic form x squared plus 4xy plus 5y squared. So, this is even easier, you just complete the squares, you will get x plus 2y the whole squared minus 4y squared plus 5y squared, which is x plus y the whole squared plus y squared and this is positive definite.

That is obvious because it is a sum of 2 squares no negative sign is occurring. So, irrespective of what x and y are, if they are not 0 this quadratic form is definitely going to be positive quantity, ok.

This concludes the proof and some examples of completing the squares. This is a course on Real Analysis, and you have just watch the video on completing the squares.