Real Analysis II
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Lecture - 22.1
Maxima and Minima in Several Variables

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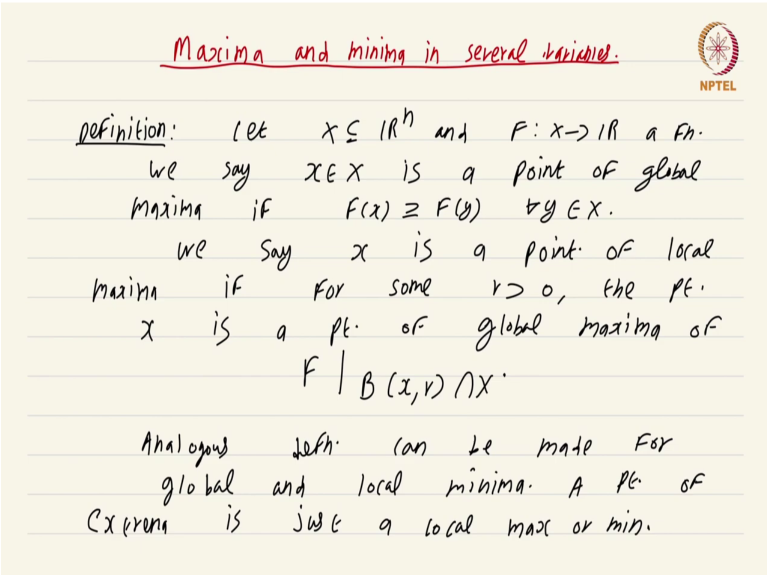

Our goal is to now study Maxima and Minima for functions defined on several variables. I will assume that you have already studied the computational aspects in a basic course on multivariable calculus and now, focus on the theory. We will justify the methods like Lagrange multipliers and the conditions on the Hessian matrix that you have no doubt seen already in the context of 2 or 3 variables.

We begin with the definition of maxima and minima, the definition works in quite a general setting. So, I will put it in a general equation setting. Let X subset of R n and f from X to R a function. We say x in X is a point of global maxima; of global maxima, if f of x is greater than or equal to f of y for all y in X. The definition is literally translating English into mathematics.

We say x is a point of global, not global local maxima if for some r greater than 0, the point a, sorry not a; the point x is a point of global maxima of f restricted to B x r intersection X. So, instead of focusing on the whole of x, I just focused near the vicinity of this point x.

You must be able to find a small enough vicinity of the point x such that x is a point of global maxima in that vicinity. Then, we say that x is a point of local maxima for f ok. Analogous definitions can be made; analogous definitions can be made for global and local minima exactly similar definitions can be made and I am going to leave that to you.

Now, you have you would have already solved an exercise in the previous set of lectures, when we studied differentiation in R n that at a point of extrema, that is a point of maxima or minima. So, let us just make that precise a point of extrema, a point of extrema is just a local maximum or minimum.

So, local maximum or minimum such points are called points of extremum or extrema. So, at an extremum, we already know that the gradient vanishes that was left as an exercise for you. Since for completeness sake, I am going to prove it now again.

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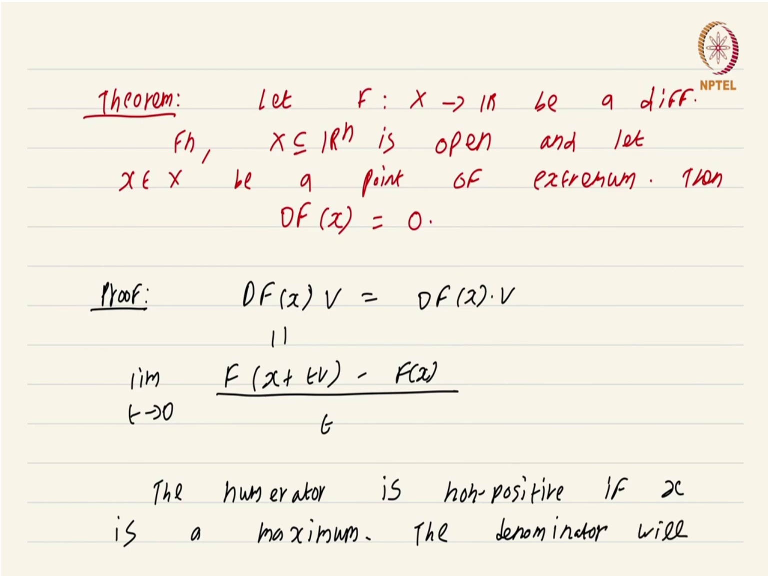

So, theorem; let f from X to R be a differentiable function differentiable function, where X subset of R n is open and let x in X be a point of extremum that is its either a local maximum or a local minimum. Then, the gradient of f at this point x is 0. Proof; so, I am just going to show, where all directional derivatives vanish and therefore, the gradient also vanishes.

We know that the directional derivative is given by Df a V. Df x V; Df x V is nothing but gradient of f x dot V and this is by definition equal to f of x plus t V minus f of x by t; limit t going to 0 right. You just look the derivative in the direction determined by V that is nothing but the directional derivative.

Now, the numerator, is negative if or rather better way to put it is numerator is non-positive, if x is a maximum right. At a point of maximum, when you are close enough to the point x, the values of the function will be less than or equal to the value of the function at x. Therefore, the numerator will be non-positive. This will be true for t sufficiently small ok.

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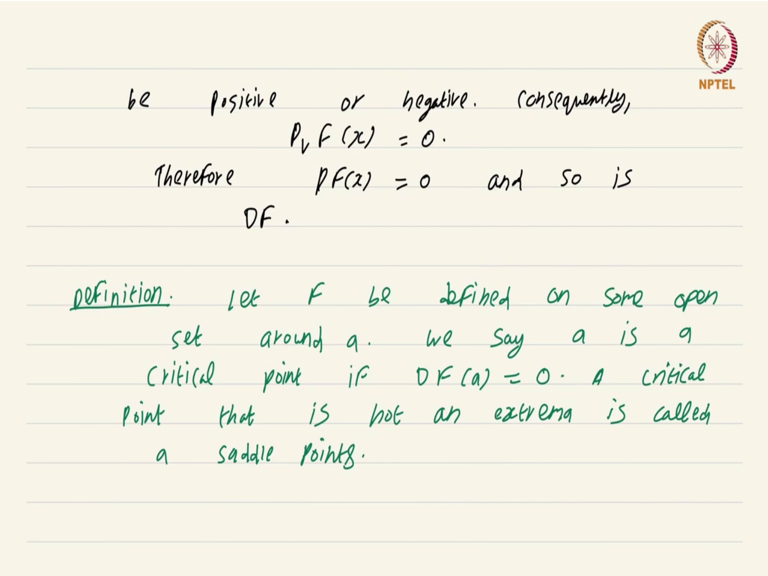

But the denominator, the denominator will either be positive or negative depending on the direction of approach. Denominator will be positive or negative right; if you are approaching from the right, it is going to be positive; if you are approaching from the left, it is going to be negative.

Consequently, D v f a f x is 0 and this is true for all vectors; therefore, D v rather D f x is 0 and so, is great and so, is graph ok. So, this concludes the proof. It is rather trivial. Now, we are going to define critical points to be those points which could possibly be maxima or minima. So, definition, let f be defined on some open set; I do not really care what open set it is, open set around a.

We say a is a critical point if gradient of f at a is 0 ok. A critical point, a critical point that is not an extrema that is neither a local maximum or a local minimum is called a saddle point, is called a saddle point. So, saddle points are points at which the derivatives vanish; but nevertheless, it happens that it is neither a point of maximum or a point of minimum local at least.

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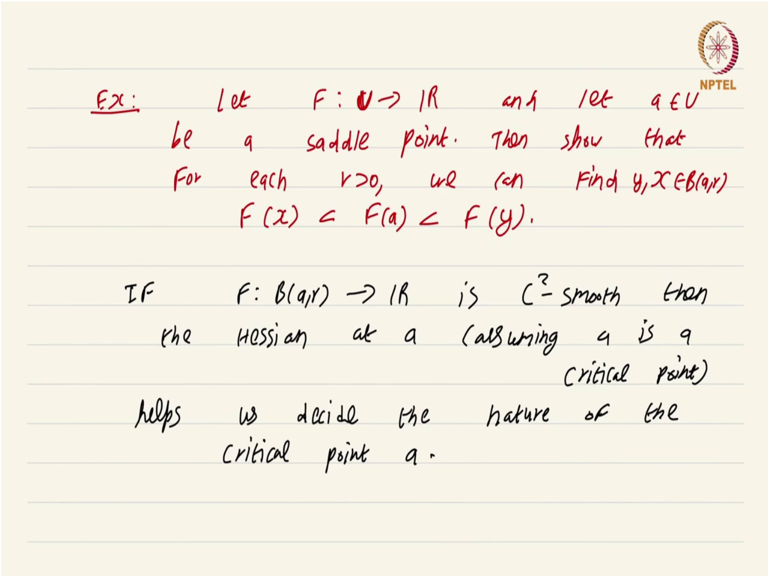

Now, how do you characterize these saddle points? Well, the next exercise that is left for you does it for you it is rather straightforward unwinding of the definition. Let f from some u to R and let a in U be a saddle point be a saddle point. So, this is a point which is neither a local maximum or a local minimum of the point a ok.

Then, show that for each r greater than 0, each r greater than 0, we can find we can find x in B a r, I mean I can say find y comma x in B a r such that f of x is less than f of a is less than f of y and this should be true for each r greater than 0, for every r in the open ball B a r, you should be able to find points x and y such that f of x is less than f of a is less than f of y ok.

Now, our goal is to determine whether a critical point is a local maximum, a local minimum or a saddle point and if possible, we would like to solve this problem using only the data about that function at the point a. Now, it turns out that this problem is solvable to a reasonable extent, once you assume f is c 2 ok.

So, if f is c 2, if f from B a r to r is c 2 smooth is c 2 smooth, then the Hessian at a, assuming a is a critical point of course; assuming a is a critical point helps us decide helps us decide the nature of the critical point that is whether it is a its a local maximum, a local minimum or a saddle point ok.

Now, we will see in the next video that associated to the Hessian, there is something called a quadratic form and whether the quadratic form is going to be positive definite or negative definite or semi definite or not semi definite or indefinite that is going to determine whether, it is going to be a local maximum minimum or a saddle point we will see that in the next video. This is a course on Real Analysis and you have just watched the video on Maxima and Minima in Several Variables.