Real Analysis - II
Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 21.2
The Tangent Space to a Manifold

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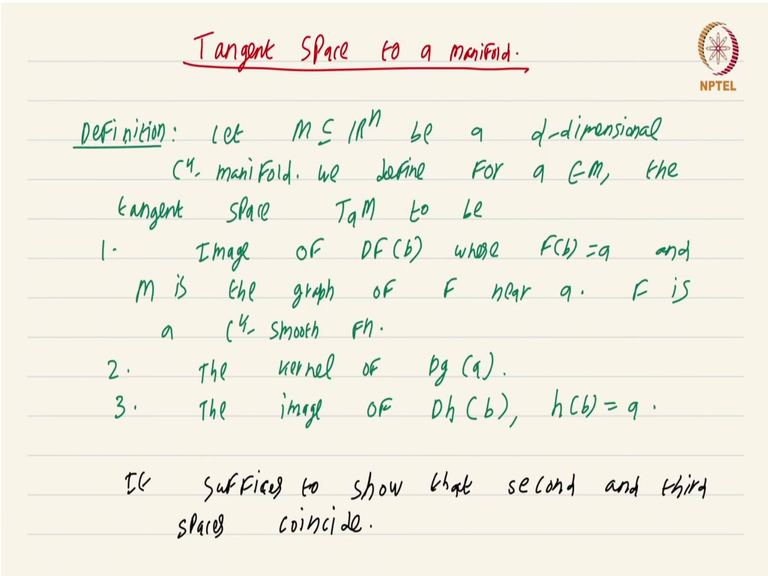

We have already seen what the tangent space to a hypersurface is. Now, we are going to generalize the same notion to an arbitrary manifold. As I had remarked when we solved or rather proved that the tangent space to a hypersurface actually coincides with our intuitive visualization of such a tangent space.

I had remarked that time that please understand this lemma carefully, it will be very useful in the study of manifolds. Well, the time has come, where that understanding is going to be called. The three definitions of manifolds will give rise to three ways of thinking about the tangent space.

We will define three vector spaces using the three definitions and will turn out that all three vector spaces are exactly the same and they confirmed to our intuitive idea of what the tangent space should look like. So, let us state the three definitions of the tangent space.

Definition, let M subset of R n be a d dimensional d-dimensional c k manifold. We define for a in M, the tangent space the tangent space TaM to be number 1, when we view the function sorry the manifold as locally the graph of a C k smooth function. Then, we define it to be just the image of D f b, where f of b is equal to a and M is the graph of graph of f near a ok and of course, f is a C k smooth function.

So, I am more or less borrowing the notation from the definition of a manifold and the first definition was that it is locally the graph. So, if its locally the graph, just look at the image of the linear map D f b. Note that map we already know is going to be of rank d. So, the image of D f b will be a d-dimensional vector space.

The second definition is when we considered the manifold as locally the level set of a C k smooth function and I am just going to not bore you with notation once again, I am going just going to say the Kernel of D g a. And to understand what this g is please refer back to the definition of the manifold, I am just borrowing the notation from the definition of the manifold ok.

And third is when it was locally parametrised near a and there, it is just the image of d h of b, where h of b is equal to a ok. So, these three are the three ways of representing the tangent space.

Now again, the third definition will give rise to a d-dimensional vector space simply because D h is also of rank d and the kernel of D g a that is also going to give you a d-dimensional manifold, I mean d-dimensional vector space. Simply because D g was rank n minus d; the rank nullity theorem will tell you that the kernel will have to be of dimension d.

So, thankfully, at least all these three definitions will give rise to spaces that are at least same dimension. Now, we are going to show that not only are the spaces the same dimension, they are exactly the same. Not only that, notice that there is a choice involved ok.

Now, there is no such result that says that near a, the manifold is the level set of a unique function. I could have chosen some other function g for which the manifold you locally near a, it is the level set of that function also that is possible. And similarly, you can have many many parametrisations of a manifold. So, it is not even clear that when you make a different choice of these spaces, of these functions you get the same spaces that is not clear either.

So, to make sure that this definition is well-defined I have to show two things for a given choice of f g and h, they all should coincide and if you change also you should still get the same space ok. Now, I am going to leave it to you to think about why it suffices to show that let me just write that down, it suffices to show, it suffices to show that the second and third space coincide; third spaces coincide.

If I show that all of this is taken care of and everything else is automatic is my claim. This is something that you should ponder about why this is true. I am just going to show that the second space and the third space coincide and everything is taken care of ok.

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Observe that $g \circ h : U \to \mathbb{R}^{n-d}$ is the zero map.

$h(U) \subseteq$ domain of g.

$Dg(a) \circ Dh(b) = 0$ transformation

image of $Dh(b) \subseteq \ker Dg(a)$.
both are d-dimensional and therefore must coincide.

Now, observe to show that the second and third space coincide, observe that g o h is a map from U, again I am borrowing the notation from the definition of a manifold; g o h is a map from U to R n minus d that is that is precisely that is the 0 map. It is precisely the 0 map. So, no need to say this ok. Observe that g circle h from U to R n minus d is the 0 map.

Why is it the 0 map? Well, simply because g maps on sorry h maps on to a piece of a manifold and g takes that piece to 0 ok. So, what I am assuming? I am technically assuming that h of U is contained in the domain of g which I can do by continuity of h and just shrinking U, if necessary. I can assume that h of U is fully contained in the domain of g ok.

Now, what is the chain rule say? It says that D g a, D h b is the 0 map is the 0 transformation; the 0 linear transformation. Well that just means that image of D h b is contained in the kernel of D g a. It is precisely under this scenario can will it happen that D h a composed or multiplied by whatever let us put a circle to make sure there is no ambiguity.

D g a composed with D h b is identically the 0 transformation that will happen precisely when image of D h b is contained in kernel of D g a. But both are d dimensional as I remarked at the beginning of this proof; both are d dimensional and therefore, must coincide and therefore, must coincide. So, this takes care of the proof; I still leave it you to ponder about why everything else is automatic ok.

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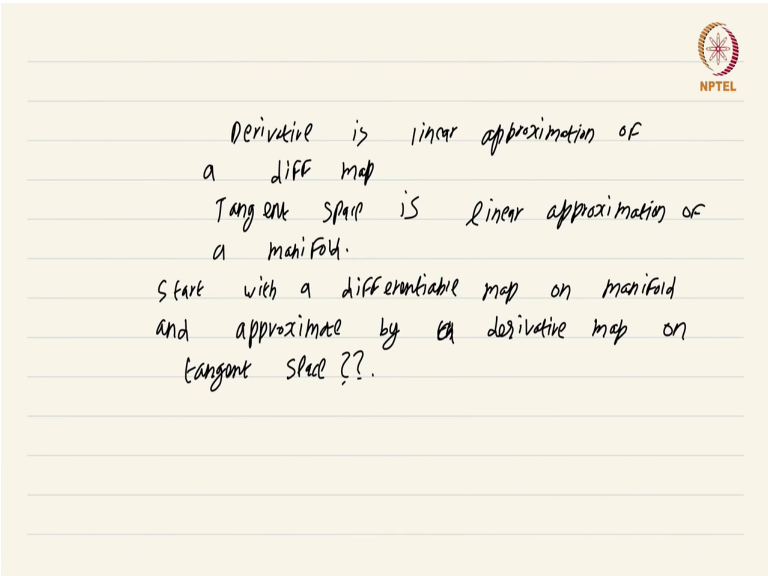

So, now, let us move on and ponder about what is it that we have done? We first studied differentiable maps from open sets in R n to R m and we saw that these are supposed to be the derivative maps or supposed to be the best linear approximation of such a differentiable map.

Now, we have approximated the manifold locally by the tangent space, just like a manifold just like a map is approximated by the derivative which is linear, the manifold which is in general a curved object is approximated by this linear subspace which is the tangent space at this point.

So, it is natural to wonder that if you have a differentiable map from a manifold, is it automatically true that that differentiable map is going to give rise to a derivative that approximates this map; but approximates it, where approximates it on the tangent space.

So, we have derivative is linear approximation of map of a differentiable map, we have tangent space tangent space is linear approximation linear approximation of a manifold can we combine these two and start with a start with a differentiable map on a manifold differentiable map on manifold and approximate by a derivative map on the tangent space, on tangent space. So, I am just combining both; is it possible to do this? Well, yes, we can do this. So, that is the next definition.

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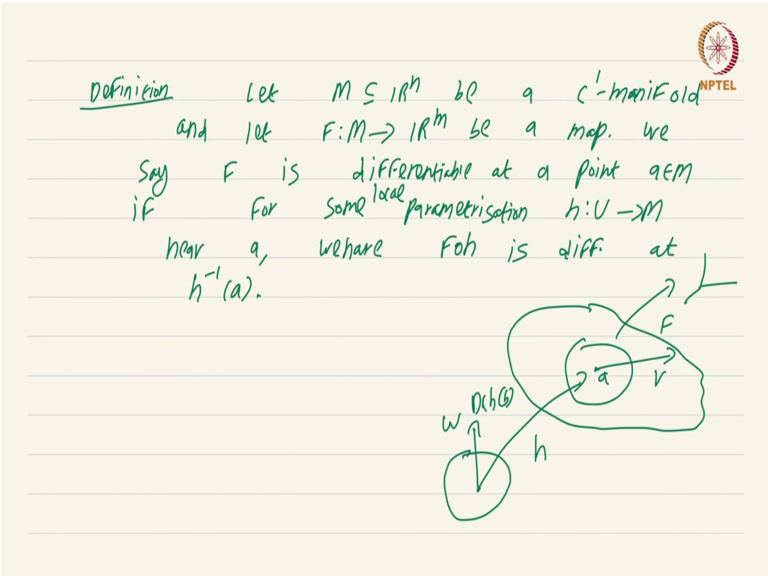

Definition, let M subset of R n be a C 1 manifold, be a C 1 manifold and let f from M to R m be a map ok. We say we say f is differentiable, differentiable at a point at a point a in M, if for some parametrisation h from U to m. So, some local parametrisation near a, we have f circle h is a differentiable at the point h inverse a ok.

So, essentially, the picture is as follows. You have a curved manifold and you have an open set that parametrises a piece of this curved manifold. You have a map from f to some other Euclidean space ok. So, I am just let me draw it properly. To another euclidean space, I have this map f to say that f is differentiable at a, you just sort of pass to an open set in R d via this map h and check differentiability at this point which maps to a ok.

So, this is perfectly a natural definition. The manifold is parametrised by a C 1 smooth function pass to a euclidean space, open set in a euclidean space via this C 1 smooth function and define differentiability there ok. So, this tells you when a function is differentiable; but this does not tell you what the derivative map is that is the next part of this definition.

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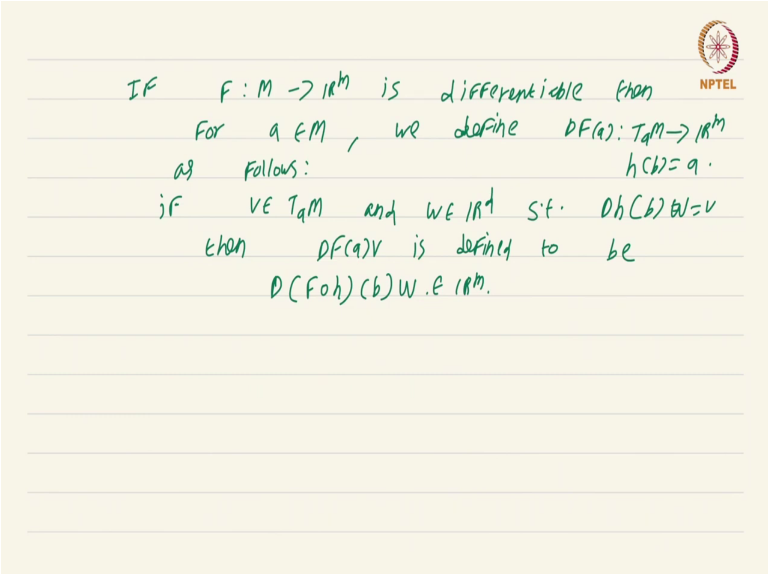

If f from M to R m is differentiable is differentiable, then for a point a in M, we define D f a from TaM to R m as follows. If V is an element in the tangent space and W is a vector in R d such that D h of b at W is equal to V. So, I am assuming h of b is the point b is the point that maps on to a under the parametrisation.

So, if you consider the parametrisation and you know that the tangent space is the image of D h at b. So, given a vector V in the tangent space, you can find a vector W in R d such that D h b V equal to W; sorry D h b W equal to V.

So, you make these choices, then D f a V is defined to be D f circle h at b of W ok. So, this will be an element of R m ok. So, this is the definition of the derivative map. Again, the basic idea is you just pass on; so, this vector V is sort of let us draw an inaccurate picture.

So, let us say this vector V is like this. Well, you just look at that vector W that maps onto this by D h of b; D h b and then, see where this vector W goes under the composition f circle h take the derivative map and see where it goes, that is the image.

So, again, there are some issues of well-definedness under this definition. There is no unique parametrisation, there could be dozens. It is not clear that this map that we have defined D f a is not clear that it is linear as well.

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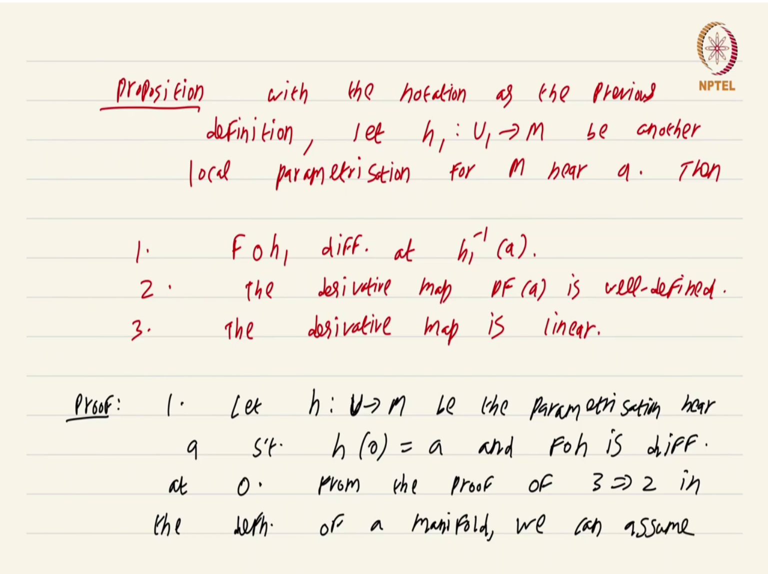

So, we have a proposition that clarifies all of this in a neat way. Proposition with the notation as the previous definition notation as the previous definition ok, let h 1 from let us say U 1 to M be another local parametrisation; parametrisation for M near a; then, we have three facts. Number 1, f circle h 1 is differentiable at h 1 inverse of a.

So, in the definition of differentiability, all it says is therefore some local parametrisation this happens that f circle h is differentiable at h inverse a. It is not clear that if I had chosen a different parametrisation, f circle h 1 at h 1 inverse b h 1 inverse a, it is not clear that it is going to be differentiable that needs to be asserted.

So, this definition is sort of independent of parametrisation. So, we want that we want the definition of differentiability to depend only on the manifold and not a particular choice of parametrisation and in fact, our definition does indeed is indeed independent of the parametrisation; the same.

If its differentiable under one parametrisation, its differentiable under any. Second, second point, the derivative map the derivative map D f a is well-defined; that means, if you are chosen a different parametrisation, you won't end up with a different derivative map and finally, the derivative map is linear. The derivative map is linear ok.

So, the proof involves one idea which requires you to now go back and watch the video about the definition of a manifold ok. So, let us prove all of this. I will mention what is it that you must check from that definition of a manifold video. Please go check that or check the lecture notes for this, I have made a remark about this.

Let h from U to M be the parametrisation, parametrisation near a such that h of 0, h of 0 is equal to a and f circle h is differentiable at 0 ok. Now, if you look through the definition of a manifold, the proof that all three definitions in fact give rise to the same I mean each definition is equivalent to the other.

At one stage, we go from a parametrisation to a local level set and the way, we did that was we considered this map h which is supposed to be a parametrisation from R d to R n and we tagged on some extra variables to make it a map from R n to R n and argued that map is going to be the derivative is going to be invertible and you applied the inverse function theorem to get an inverse ok.

So, using that inverse using that inverse, you can actually say something about this parametrisation. What is it that we can say? Well, from the proof, from the proof of 3 implies 2 in the definition of a manifold in the definition of a manifold, we can assume; we can assume; we can assume that we can find; that we can find a C 1 smooth map; a C 1 smooth map which we call h inverse for simplicity.

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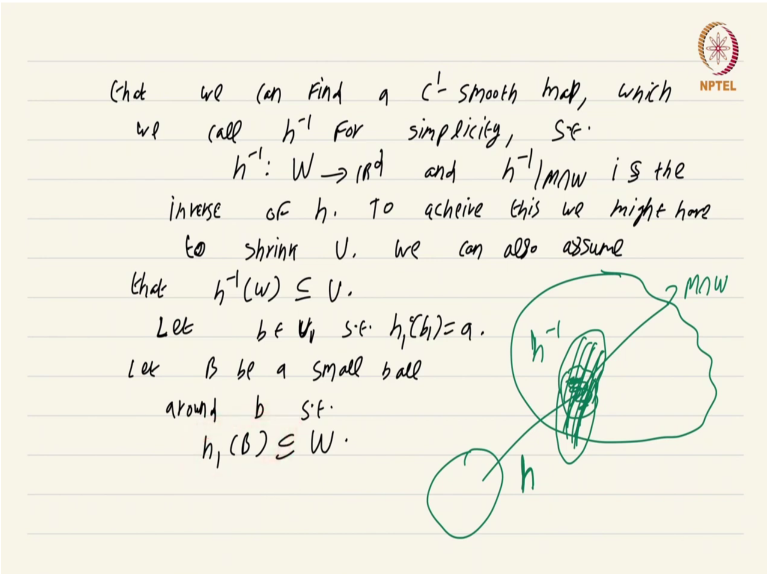

You will understand why I am using this notation h inverse ok, such that h inverse is from W to R d and h inverse restricted to M intersection W is the inverse; is the inverse of h ok. So, again, let us draw a bad picture to have good understanding. We have a manifold, we have an open set and we have this h that parametrizes this. We know that h inverse is a continuous thing and it is this is supposed to be M intersect W. W is an open set which is larger, it is in R n ok.

So, what this says is you can find a map h inverse which is defined on this W which is a C 1 smooth map ok and h 1 inverse maps W to R d and when you restrict h 1 inverse to M intersection W, it is a sorry not h 1 inverse, h inverse. When you restrict h inverse to M intersection W, it is indeed the inverse of h ok.

So, what this achieves for us is h inverse was a priori just a map defined on a piece of a manifold. We cannot really talk about smoothness of a map that is not defined on an open set this extension allows us to talk about smoothness of h inverse. The fact that you can extend h inverse which is just defined on a small piece of the manifold to an open set in R n allows us to talk about the differentiability of the inverse map ok.

So, please check the proof of the fact that 3 implies 2 in the definition of a manifold to see to understand why this is true ok. Now, to achieve this actually I must this thing to achieve this, we may have to shrink U ok. That is because we have to apply the inverse function theorem and all that and inverse function theorem is a local result.

So, you might not be able to get the map h that is we had x we had first of all made this map h into a map defined on R n and then, inverse function theorem will give you a small neighbourhood. It is not clear how big that neighbourhood is ok. So, you may have to shrink U. Also, look through the remark in the notes following the three definitions of manifold, I have talked about this at some detail ok.

Anyway this is not so difficult, it will be a good exercise for you to work this out, why this is true ok. We can also assume; we can also assume that h inverse of W is contained in U. Just shrink W, if this does not happen; h inverse is anyway continuous, so you can do this ok.

Now, let b be in U 1 such that h of h 1 of b 1 is also a. Remember h 1 was another parametrisation and it will be defined in a different open set U 1 that might not even contain 0, but there will be a point that maps b that maps on to a, that point we are calling it b 1 ok.

Now, let B be a small ball around B such that this h 1 of B also maps to this W ok. This W neighbourhood comes from the definition of the manifold with respect to this parametrisation h ok. Again, look through the notation in the definition of a manifold to understand what this W is. There will be a W corresponding to h 1 also; but that could be very different, but again by continuity, I can just find a ball such that h 1 B is contained in W ok.

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Then

$$f \circ h_1|_B = (f \circ h) \circ (h^{-1} \circ h_1|_B).$$

Observe that $h^{-1} \circ h_1|_B (b) = 0$.
and this composition is Diff. at b.

$f \circ h_1|_B$ is diff. at b. $f \circ h_1$ is diff. at 0.

Now, here the next equation is the crux of the proof, then f composed with h 1 restricted to B is nothing but f circle h inverse circle h 1 restricted to B ok. So, the composition of f and h 1 restricted to B is nothing but composing f and h, then composing that with h in h inverse composed with h 1.

Essentially, these two just cancel off; these two just cancel off ok. This is completely elementary, just h and h inverse just cancel off. So, you get f composed with h 1 restricted to B is nothing but f composed with h composed with h inverse composed with h 1 restricted to B ok.

Now, observe that observe that the way we have set up things h inverse compose with h 1 restricted to B of b is nothing but 0. Why is that the case? Because h 1 takes b to a and h inverse will take 0 a back on to 0 ok. So, this will show that h inverse composed with h 1 restricted to B of b is nothing but 0 ok and this composition is smooth composition is C 1 smooth.

Why? Because this map this map is C 1 smooth at the point 0 and this not ah. So, this sorry about that, this composition is not C 1 smooth. This composition is differentiable is differentiable at b; why is it differentiable at b? Because this is differentiable at 0 by hypothesis and this is certainly going to be differentiable at all points because both maps are actually C 1 smooth.

So, what this shows is that f composed with h 1 restricted to B is differentiable at b. Now, differentiability is a local property and therefore, you can conclude that f composed with h 1 is differentiable at 0 excellent. So, the definition of differentiability does not depend on the choice of parametrisation.

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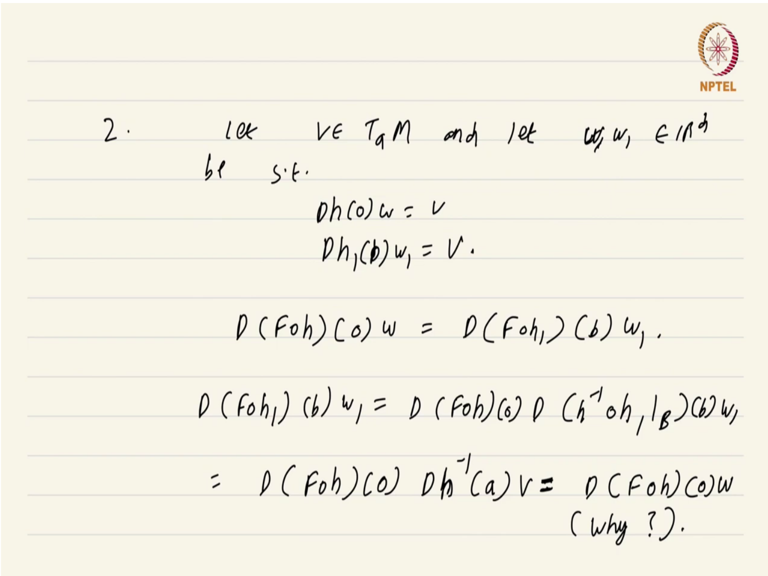

Now, on to the second part, the second part asserts that the definition of the tangent space the tangent map, the derivative map from the tangent space is well-defined ok. So, let V be in the tangent space and let W W comma W 1 in R d be such that be such that D h of 0 of W is V and D h 1 of b of W 1 is V ok.

Now, the goal is to show that D f circle h at the point 0 of W is equal to D f circle h 1 at the point b of W 1 so that both images are the same. Therefore, both derivative maps defined using the parametrisation h and the derivative map defined using the parametrisation h 1 are both the same ok.

Now, what we are going to do is we are going to pull the same trick D f composed with h 1 of b of W 1 is just D f circle h f composed with h at 0, then D h inverse composed with h 1 restricted to B at the point b of W 1. This just comes immediately from the first part and this is nothing but D of f composed with h at 0, then D h inverse h inverse at the point a at the point a of D h 1 at the point b of W which is just V ok.

So, the final composition, this final composition, I am applying the chain rule and writing D h inverse at the point a, then D h 1 of at the point b of W 1, but D h 1 of at the point b of W 1 is just V ok. Now, this is nothing but this is nothing but this is nothing but D f circle h at the point 0 at the point 0 of W and I want you to check why this is true; I want you to check why this is true that D f circle h at 0 of W is equal to this expression here ok.

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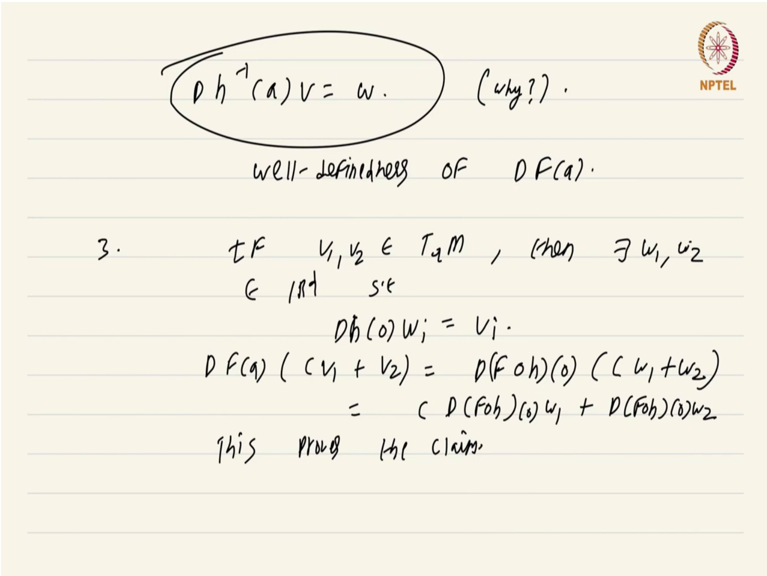

So, one part I mean the assumption that I am making is D h inverse a sorry D h inverse 0 sorry D h inverse a of V is W. This is what you have to justify ok. So, this concludes the proof of well-definedness of the map D f a.

Finally, we have to show the linearity. For linearity, if V 1 comma V 2 are in the tangent space at M ,then there exists W 1, W 2 in R d such that D h of 0 of W i is V i ok. Then, D f at the point a of C W 1, sorry C V 1 plus V 2 is nothing but D f circle h at 0 of C W 1 plus W 2 because under the map D at 0, C W 1 plus W 2 will get mapped on to C V 1 plus V 2.

And this is nothing but C D f composed with h 0 of W 1 plus D f circle h composed with 0 of W 2 and this proves the claim; this proves the claim, linearity follows ok. So, this shows that the derivative map is not only well-defined, it is also linear.

So, this concludes this video on Tangent spaces. Please do check out the exercises, where we relate this tangent spaces to curves which are natural and that will also give you strengthen your intuition as to why this is the correct definition of the tangent space. Whenever you are dealing with abstract stuff, you have to understand that the abstraction coincides with our intuition and these exercises will do that for you.

This is a course on Real Analysis and you have just watch the video on the Tangent space to a manifold.