Real Analysis - II
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Lecture - 21.1
Examples and Non-Examples of Manifolds

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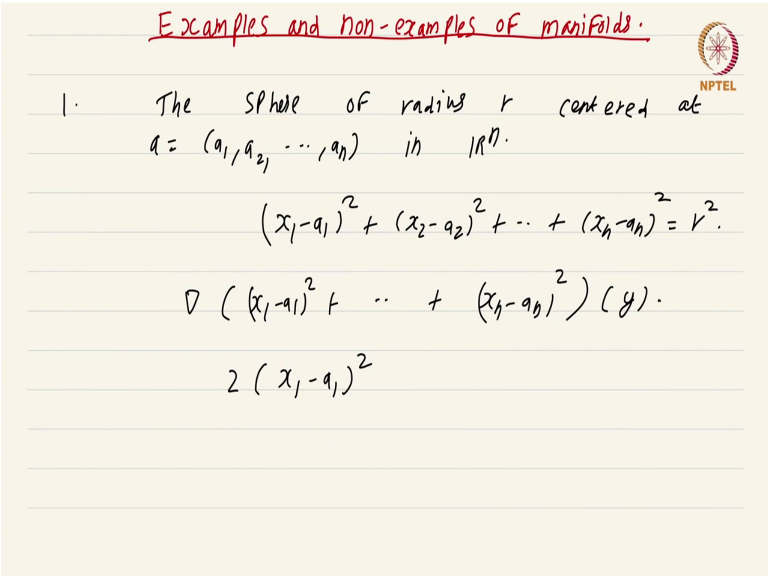

The definitions of manifolds that we have given is admittedly quite abstract. Let us have a good understanding of the of these definitions by seeing several examples and non-examples of the type of objects that can be manifolds. Example number 1, we already saw this example, a special case of this before we prove the implicit function theorem.

This is the example of the unit sphere in r 3. More generally, we can study the sphere of any radius or rather let us fix a sphere of radius r centered at the point a equal to a 1 to a n in R n. So, this is the sphere of radius r centered at the vector a 1, a 2 dot dot dot a n in R n. So, the equation of such a sphere is easily seen to be x 1 minus a 1 squared plus x 2 minus a 2 squared plus dot dot dot x n minus a n squared equal to r squared.

This is from basic analytic geometry. You know that this is the equation of the sphere centered at the point a 1, a 2 dot dot dot a n of radius r. Now, this is a manifold simply because look at the gradient, look at the gradient of this function x 1 minus a 1 squared plus dot dot dot x n minus a n squared at some point y. Let us say, now this is just going to be 2 x 1 minus a 1 squared rather let us write y as y 1 to y n.

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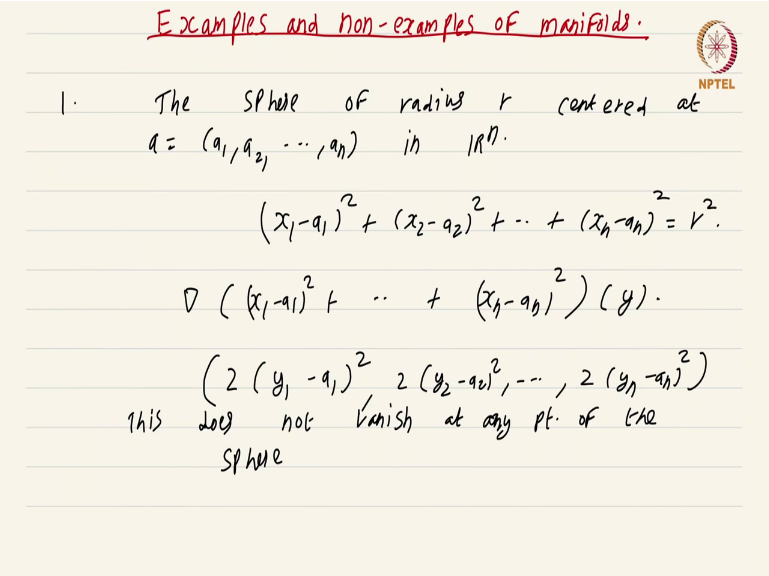

And we can just say this is going to be 2 y 1 minus a 1 squared comma 2 y 2 minus a 2 squared comma dot dot dot 2 y n minus a n squared. This is going to be the gradient vector and this does not vanish. This does not vanish at any point of the sphere. So, minor typo; I differentiated incorrectly. This is not square, this the 2 will disappear when I differentiate ok.

So, it is very clear to see that the sphere of radius r centered at the point a 1, a 2 dot dot dot a n in R n is a manifold. Because at every point that satisfies this equation, we see that the gradient is going to be this, which clearly does not vanish at any point that satisfies this equation. So, the sphere is a manifold. This is in fact, a C infinity manifold ok.

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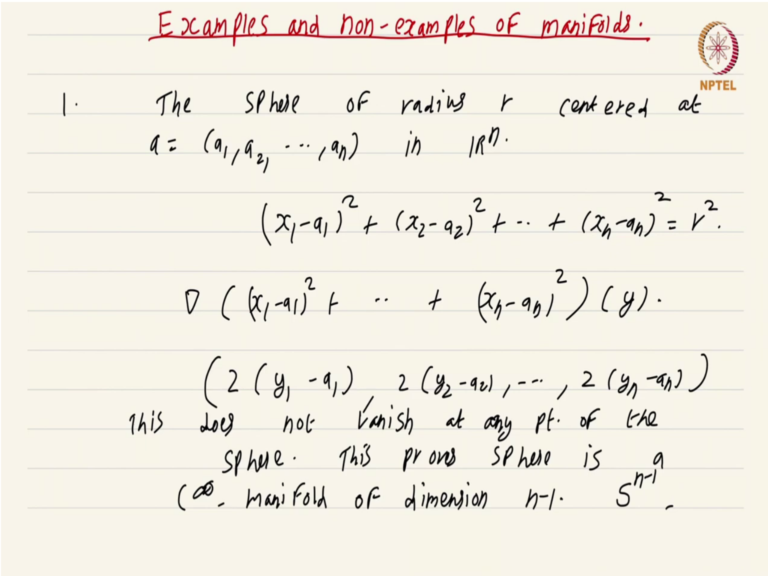

So, this proves, sphere is a C infinity manifold of dimension n minus 1. So, the classical notation for the sphere of dimension n minus 1 is actually S n minus 1 ok. So, this n minus 1 denotes the dimension.

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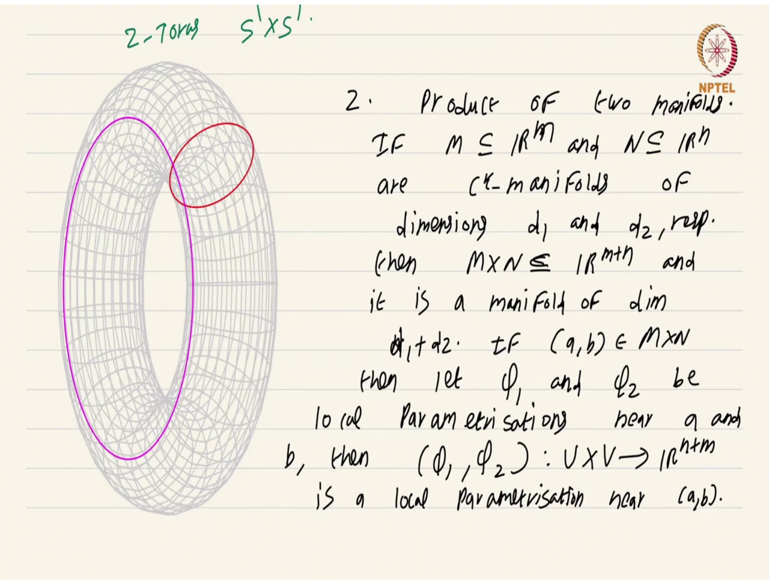

So, that was the 1st example. The 2nd example, we are going to see is of this object that you see in front of you this is a torus. This is a torus. So, this is going to be a product; this is actually going to be a product. So, let us just first consider the product of 2 manifolds and then, specialize to this torus. So, product of 2 manifolds.

This is also a manifold. Well, if M subset of R n and N subset of sorry let me make this R m and N subset of R n are C k manifolds C k manifolds of dimension d 1 and d 2 respectively; d 2 respectively. Then, this product M cross N, this is a subset of R m plus n and it is a manifold, it is a manifold of dimension m d 1 plus d 2.

Why is this the case? Well, if a comma b is a point in M cross N, then let phi 1 and phi 2 be local parametrisations; local parametrisations near a and b, near a and b. Then, this new function phi 1 comma phi 2, so let us say phi 1s domain was U and phi 2s domain was V. So, this will be from U cross V to R n plus 1 m is a parametrisation, is a local parametrisation near a comma b ok.

So, you just take two parametrisations and just sort of take their product and you will get a parametrisation of the product manifold ok. So, this immediately follows, this immediately follows that this product of two manifolds is a manifold. So, this object that I have drawn here is called a torus. It is looks like a tire or a tube or a donut or whatever. So, this is actually supposed to be the product manifold.

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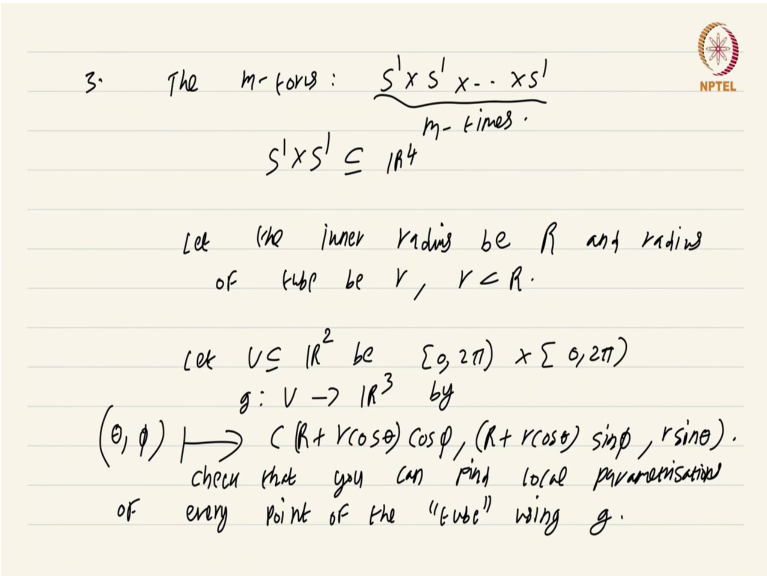

So, let us take example number 3, the m-torus this is S 1 cross S 1 dot dot dot cross S 1 m times. So, repeating the argument in the previous part, we will get that this m torus is also a manifold of dimension exactly m because we are taking m copies of S 1 and taking the Cartesian product.

Now, what is this picture, I have drawn here? Well, this is the 2-torus. This is the picture of the 2-torus. This is actually supposed to be S 1 cross S 1. So, the way to imagine it is you are taking one copy of S 1, you are taking 1 copy of S 1 and just sort of revolving the other copy of S 1 around this one copy of S 1.

Now, if you think about it for more than a few minutes, you will realize that this S 1 cross S 1 is a subset of R 4 and what we have drawn here is clearly a subset of R 3. So, in what sense is this the 2-torus? Well, for that we need to introduce the notion of a diffeomorphism and so on which we will do in due course. But for the time being, you can check that this object that we have drawn here is also a manifold.

We will later show that this manifold is diffeomorphic to S 1 cross S 1. So, this is a good representation in R 3 of the 2-torus S 1 cross S 1. So, how do you check that this tube like object is a manifold? Well, we can parametrise it. So, there are 2 radii involved. Let the inner radius be capital R and radius of tube radius of tube be small r and small r is strictly less than capital R, otherwise things will go bad. You can imagine what will happen.

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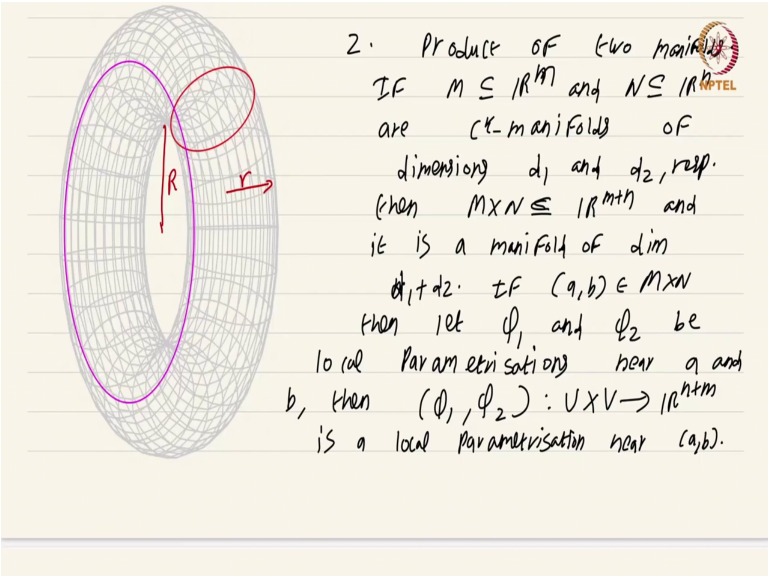

So, this inner radius is supposed to be this radius; the radius from the center to this and this is supposed to be the radius sorry about that, this is the inner radius and this is supposed to be the radius of the tube ok. So, the way I have drawn it, the inner radius looks smaller. So, let me draw it in a different way.

This is the inner radius capital R and this is small r ok. So, what is this explicit parametrisation of this tube? Well, you can write down a parametrisation which need not be a global parametrisation; but you are going to manufacture local parametrisations out of this sort of this function that I am going to give you. So, let U subset of R 2 be 0 to 2 pi cross 0 to 2 pi ok.

So, this is just a rectangle in the plane. Let me for just for convenience, let me just make it 0 to 2 pi half open ok. So, the point 2 pi is not there in these intervals. So, define g from U to R

3 by R plus small r cos theta, then cos phi ok. So, cos phi. So, this is theta comma phi map. So, I am calling the variables theta and phi. So, R plus r cos theta cos phi, then R plus r cos theta sin phi, then r sin theta ok.

Check that you can find local parametrisations of every point of the tube using g. You will have to play around with g a little bit to do this, every point of the tube using g ok. So, this function g that we have constructed that takes theta comma phi to this set of three points; R plus r cos theta cos phi, R plus r cos theta sin phi and r sin theta this is going to give you the local parametrisations that you need ok. So, I am going to leave it as an exercise to you to show that this tube is in fact a C infinity smooth manifold ok.

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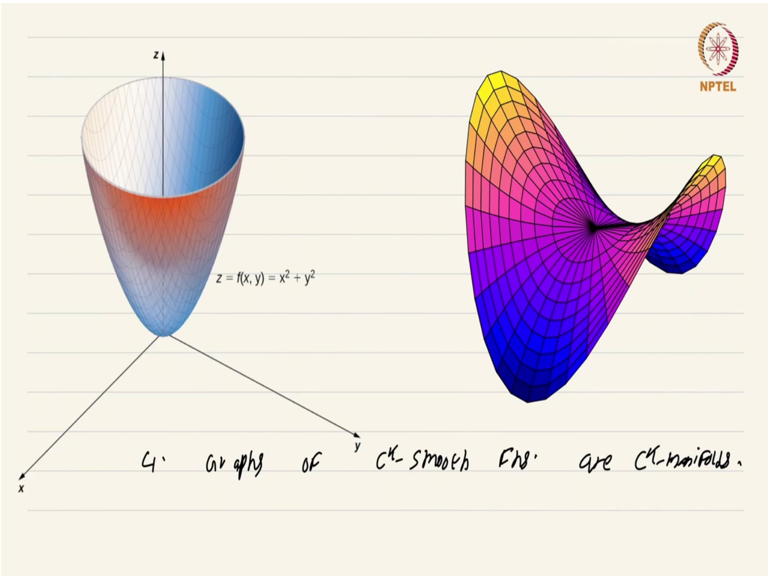

Now, the next class of examples that we are going to see are graphs; here are two graph that are displayed for you this is the graph of x squared plus y squared and this is the graph of x squared minus y squared. This is known as the paraboloid; this is known as the hyperboloid. So, these are figures that are very commonly studied. In fact, this paraboloid is used extensively in Engineering and this hyperboloid is used extensively in Architecture.

So, these are objects that arise as graphs or we know that the graph is always going to be a manifold. So, this is yet another example graphs of smooth functions, graphs of C k smooth functions. These are C infinity by the way. C k smooth functions are C k manifolds.

This is just by definition. A manifold is a more general object than a graph. So, therefore, graphs are going to be manifolds ok. Now, let us see some non-examples before we proceed any further. We want to see some situations, where the object is not going to be a manifold.

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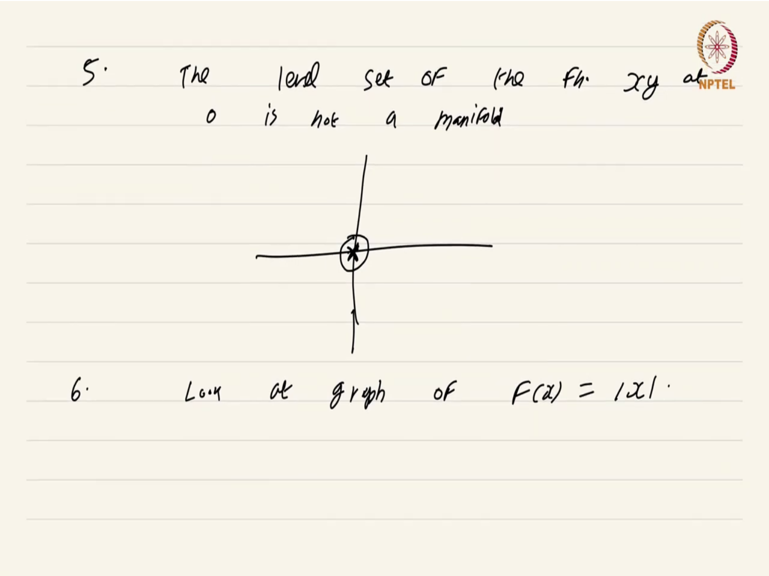

So, let us say this is the example number 5, the level set, the level set of the function xy at 0 is not a manifold ok. We have already seen this, when we studied the implicit function theorem that this object is going to be a cross and it is not going to be a manifold. More precisely, at this origin you cannot find a local parametrisation nor can you write this any neighbourhood of this you can nor can you write any neighbourhood of the neighbourhood of that as a graph.

Anyway, all three conditions are equivalent. If you cannot parametrise it, you cannot write it as a graph as well. So, the level set of the function x y at 0 is not a manifold. Let us see another example where you have a function that is actually a graph, but it is still not a manifold. Look at graph of f of x equal to mod x ok; the graph is going to look like this. The graph is going to the graph is going to look like this.

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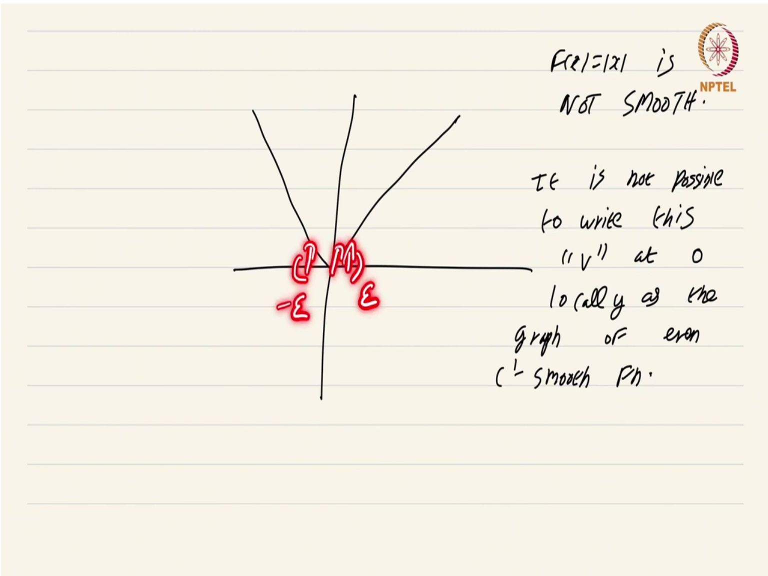

This is not a manifold, precisely because we have demanded that there is some level of smoothness involved in the definition of the manifold. All three definitions that we gave involved functions that are smooth. Now, the function f of x equal to mod x is not smooth is not smooth ok.

Now, that does not mean that it is not it is impossible that f of a this graph is a smooth manifold. The it is that is not guaranteed. It could be the case that we could write this object, this V as the graph of some other function which happens to be smooth ok. Is that possible? Well, actually not so, it will not possible; it will not possible to write this V at 0 locally as the graph as the graph of even a C 1 smooth function. It cannot be even a C 1 smooth manifold.

Well, why is that the case? Well, think about this for a second. If we could write this as a graph, let us say over some small neighbourhood minus epsilon epsilon, let us say for argument sake, what is that function going to be whose graph is going to be this V shape in this neighbourhood minus epsilon epsilon? If you think about it for a few seconds, you will realize that it has no choice, but to be the graph of mod x right.

The function is determined entirely by its values; if you are going to write this V as a graph of a function, it is going to be the graph of mod x ok. So, I want you to ponder over this simple argument to show that it is not possible to write this near this point 0 as the graph of a C 1 smooth function.

The function if at all there is such a function is forced to be mod x itself. So, this is impossible ok. So, what we have seen is some examples and non-examples of manifolds, let us finish with one more example that comes from the level set.

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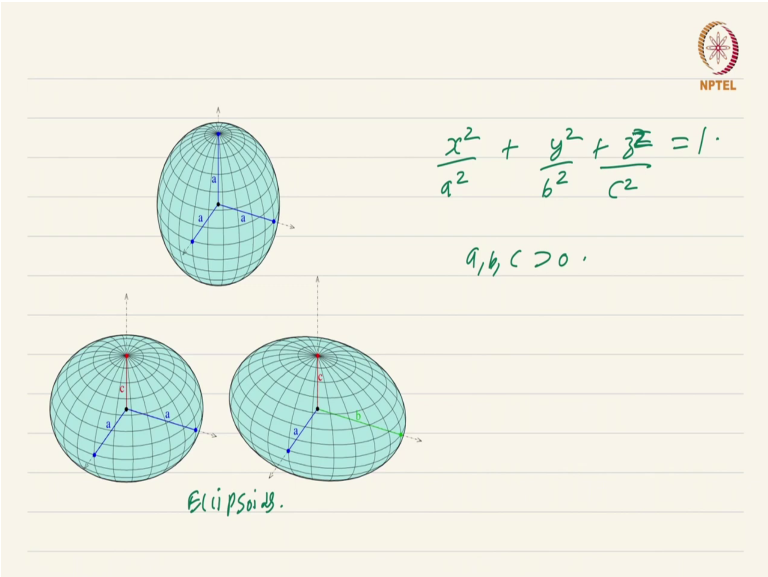

This is ellipsoids, these are ellipsoids ok. They are also graphs of functions ok. What are the what is the function that is going to give rise to these ellipsoids? Let us just say that in r 3 it is just x squared by a squared plus y squared by b squared plus z squared by c squared equal to 1 ok, where a comma b comma c are numbers greater than 0. So, x squared by a squared plus y squared by b squared plus z squared by c squared equal to 1.

These will give you these various objects, depending on the relationship between a, b and c, where you might even get a sphere which is this case when a equal to b equal to c and depending on how different they are, you might get different objects which are elongated x shaped, almost earth shaped also the earth is actually an oblate spheroid. So, you get various objects which are somewhat like oblongated balls ok.

Now, let us concentrate some our effort on understanding the relationship between the definition of the man of a manifold as the low as locally being the graph of a C k smooth function and the final definition of a manifold as an object that admits C k smooth local parametrisations; both of these seem related. What is the precise relationship between these two?

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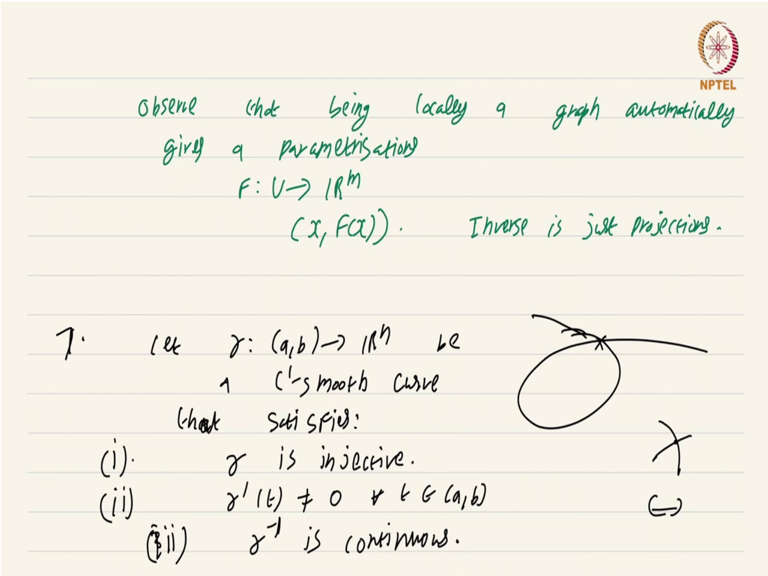

Well, first observe that being locally a graph, locally a graph automatically gives a parametrisation, automatically gives a parametrisation. No additional effort is needed. So, for instance, if f is from U to R m and you want to parametrise the graph of this function, just consider x comma f of x.

This will satisfy all the conditions needed for the parametrisation. We briefly spoke about this in the previous video. Why will it be full rank? Because of this contribution of this x, because of that it is going to be full rank. Why is the inverse going to be continuous? Because the inverse is just inverse is just projection; inverse is just projection.

So, the graph gives rise to a parametrisation, local parametrisation that has all the properties needed in definition 3 automatically; but if you in general take just a parametrisation, that is just a function that maps onto the onto a subset of the manifold, it need not satisfy the full rank condition, it need not satisfy that the inverse is continuous so on and so forth.

You need those conditions explicitly. So, in the notes, I have given several exercises to see what goes wrong, if you drop some of these conditions ok. So, there are several things, several exercises I have given, where things will look like it is going to be a manifold; but it is not.

Please work out those exercises into get a good idea of what exactly goes wrong in some cases, where an object is almost a manifold; but because 1 or 2 conditions are not satisfied, things go horribly wrong ok. So, this is a huge bouquet of manifolds we have seen.

I am going to end with one more general example that is the surface of revolution. But before I do that, I first want to consider curves as manifolds, not all curves are manifolds. I believe this is example 7. For instance, look at this curve, this curve is not a manifold At this point; it's things are going to go horribly wrong. There is no way to write that near that vicinity to write this as a graph of a smooth function that is just impossible.

So, some restricted class of curves will indeed be manifolds ok. So, let gamma from open interval a, b to R n be a C 1 smooth curve that satisfies well I am just going to tag on all the conditions, I need to make this a manifold. So, the first condition is gamma is injective so that things like intersections and crossings do not happen and break local euclideanness.

So, precisely what goes wrong at this point is that if you zoom in at this point, it is essentially the figure is going to look something like this ok and this is not locally Euclidean you, this is does not look the same as an open interval. So, in topological terms if you know a bit of topology, these two objects are not homeomorphic to each other ok. So, that is essentially what goes wrong; it is not locally Euclidean.

Gamma prime of t is not equal to 0 for all t in a, b ok and the 3rd condition is gamma inverse is continuous. So, I have tagged along all the conditions needed to make gamma into a global parametrisation. So, such an thing is such a gamma is also called a global parametrisation of the curve. It turns out that all non-compact one dimensional manifolds will have a global parametrisation, but that I am not going to prove in this course ok.

So, this is an example curves in higher dimensions, give you examples of manifolds. In particular I believe this is the very first example of a manifold in R n which is not of dimension n minus 1. All the other manifolds examples that I have given are ok not exactly S 1 cross S 1 was two dimensional in R 4. So, I correct myself.

But this is an example of a very low dimensional manifold sitting inside R n ok. Now, you might wonder this since I restricted gamma to be from open a, b and not close to a, b; what about the circle which we know is as a manifold S 1 is that not covered?

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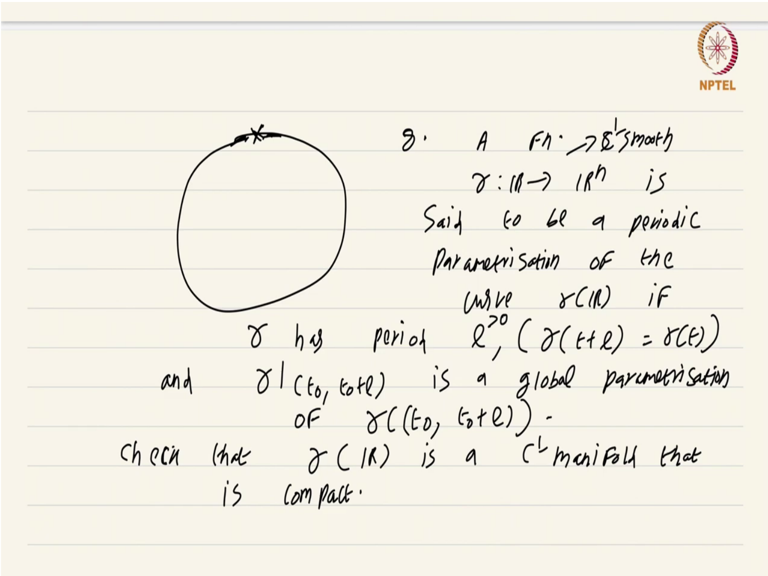

Well, that is also covered, but you cannot quite get a global parametrisation of the circle; but you can get what is called a periodic parametrisation. So, that is defined as follows. A function gamma from r to R n is said to be a periodic trans periodic parametrisation, of the curve gamma of R, if gamma has period l.

I mean that this just means gamma of t plus l is equal to gamma of t that is the meaning of the period. If gamma is period l and of course, this l should be the least number such that this happens ok, least number such that this happens and this period I am requiring it to be greater than 0. I require the period l to be greater than 0.

It is the least number such that gamma of t plus l equal to gamma of t, for all t in r and gamma restricted to t naught comma t naught plus l is a global parametrisation of gamma of t naught comma t naught plus l ok. So, you cannot globally parametrise the whole curve, but you can parametrise almost the entire thing except the one point.

So, if you remove if you remove this one point, of course, you can write this as the image of an open interval, you can do that and you can check that this will be a manifold. Even if you remove this one point, it is still going to be a manifold. Check that; that is indeed the case you might think that these two portions of the circle getting arbitrarily close to each other is going to be a problem; but it will not be. It is still going to be a manifold ok.

So, global parametrisation just means that it is a function defined from r to R n which is sorry periodic transformation means it is a function gamma from r to R n which is periodic and such that in each period interval, it is going to be a global parametrisation ok.

So, you can check, check that gamma of r is a of course, a function smooth C 1 smooth is a C 1 manifold gamma for the C 1 manifold that is compact ok. So, you can exhibit compact manifolds as images of periodic parametrisations. I want you to also find out a periodic trans parametrisation of the circle. That is rather easy to do ok.

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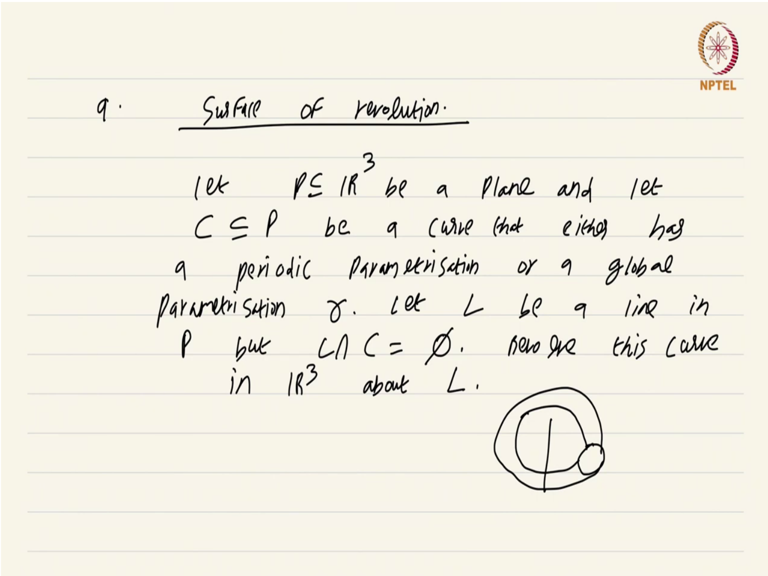

So, final example is that using these curves, I am going to define a general manifold called the surface of revolution that is you essentially take a single curve and rotate it about a central axis and you get a figure in R 3, which not R 3 in higher dimensions which happens to be a manifold ok.

Now, you can consider this torus also as a surface of a revolution. What you do is you look at this circle, you look at this circle that is here and you just you are just rotating it about this axis which goes through the center of this torus ok; center as in the center point of this torus, you are just rotating this entire circle about that single line to get the torus the tube ok.

So, let us get back to our example surface of revolution, how is this defined? Let P subset of R 3 be a plane ok. I am going to just consider surfaces of revolution in R 3 for simplicity sake and let C subset of P be a curve that either has a periodic parametrisation, or a global or a global parametrisation gamma.

So, I am considering a one dimensional manifold C that admits either a periodic parametrisation or a global parametrisation. It turns out that any one dimensional manifold always admits either a global parametrisation or a periodic parametrisation; but I am not going to prove this in this course.

So, that is usually shown in differential topology that any one dimensional manifold is either going to be either going to admit a global parametrisation, if it is non compact and a periodic parametrisation, if it is compact ok. So, let C be such a curve. Let L be a line that is also in this plane P, but L intersection in C is empty ok. So, take a line in this plane and such that L intersection C is empty ok.

Now, what I am going to do is revolve this curve in R 3 about L ok. So, what I am going to do is I am just. So, both L and this curve lie in 1 plane. So, essentially, what I am going to do is I am just going to visualize a third dimension that is perpendicular to this paper and then, lift this curve up and just rotate it about a circle so that this curve revolves around this line in three dimensional space once ok.

So, if I were to finish this picture, if this were a circle, this were a circle like this. Then, if I do complete this procedure, I would get something like this, a tube ok. I have drawn it badly, but I hope you get the idea. I will sort of get a tube like the torus that we saw I will get a tube sitting in R 3 ok. So, I am not going to prove in general that this is going to be a manifold, I am just going to take a special case, I am going to take a special case which reflects all the general ideas ok.

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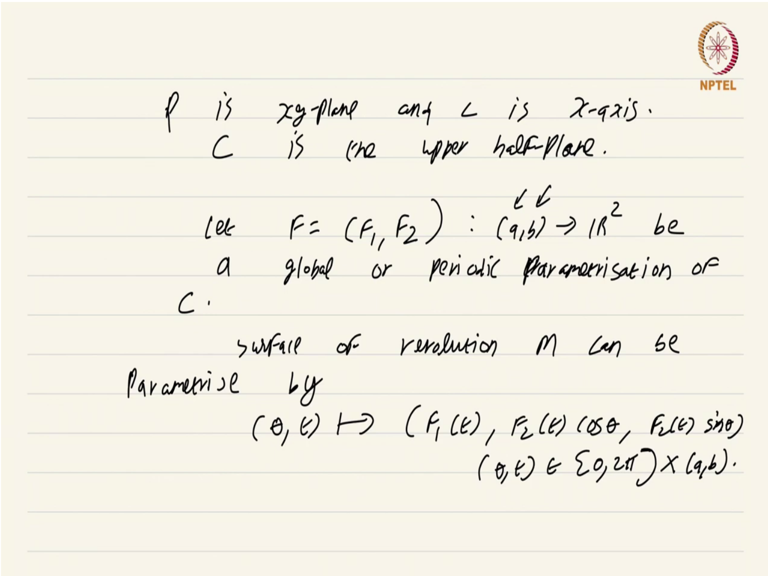

So, special cases P is xy-plane and l is x axis. Under these assumptions, I can assume that C is in the upper half plane, C is in upper half plane that is C is in the portion of the plane, where y is greater than 0 ok. Think about why I can make this assumption. I am now I am going to give you a parametrisation using which you can actually construct local parametrisations that satisfy all the conditions in the third definition of a manifold. So, this is as follows.

Let F equal to F 1, F 2 of from a b to R 2 be so be a global or periodic transforming parametrisation of so of C ok. In case this is a periodic transformation parametrisation, a and b would be minus infinity and plus infinity. Here, I just mean any open set; it really does not matter, it could be any open set.

Now, I am going to parametrise this surface of revolution m, surface of revolution m can be parametrised by can be parametrised by theta comma t, this maps to F 1 of t, F 2 of t cos theta, F 2 of t sin theta ok, where theta comma t comes from 0 to 2 pi times a b ok.

This is a global map, it will map onto the manifold m and using this map, you can construct local parametrisations of m that will satisfy all the conditions needed to show that m is a manifold.

All the conditions of the third definition of a manifold will be satisfied, you have to modify this map a little bit and I mean this will not work globally, you will have to localize this map essentially ok.

So, there are a lot of things I have left for you to do checks that is because getting your hands dirty and doing these computations and checking in fact that certain things I claimed are manifolds are indeed manifolds is probably the only way to learn this abstract notion of a manifold. This is a course on Real Analysis and you have just watch the video on examples and non-examples of manifolds.