Real Analysis - II
Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 20.1
The Definition of a Manifold

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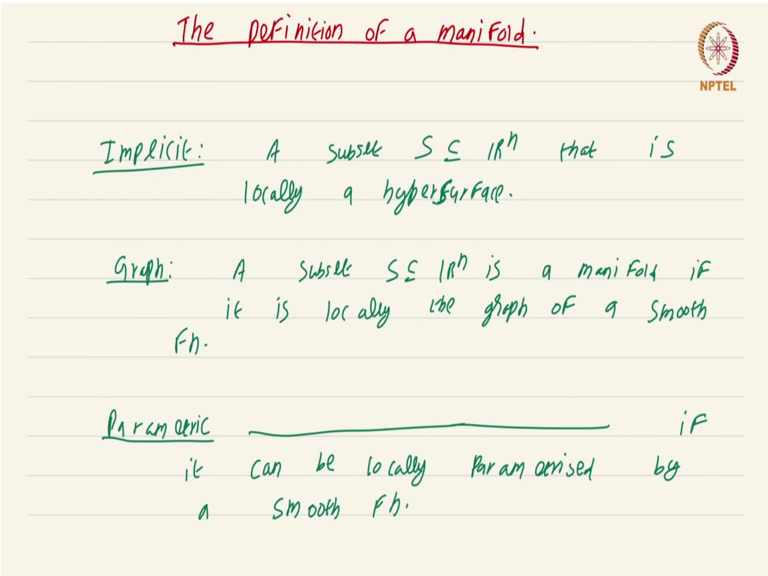

We now come to the central definition of this course The Definition of a Manifold. This notion is of central importance that we are not going to give just one definition, but in our customary style we are going to give several definitions at once and prove they are all equivalent.

Now, before we get to the definition we must first understand what is it that we are trying to formalize by a definition. Typically, we have some intuitive pictures in mind and we try to capture that in the form of a rigorous mathematical definition. So, the general definition that captures our picture is somewhat sophisticated and technical and the necessity of such a general definition will be completely opaque to you at this stage.

So, the definition of a manifold is in general a separable metric space or even a second countable or paracompact host of topological space that is locally Euclidean. Now this is a bunch of technical terms you have already come across separable metric space locally Euclidean will be fairly easy to define, but I am not sure whether you have come across second countability or para compactness.

So, the necessity of such a technical definition it will be somewhat unclear to you at this stage. So, rather than focus on this general notion of a manifold we will focus on manifolds that are subsets of R n ok. A manifold is an object that locally looks like Euclidean space what I mean by that is if you zoom into that manifold with a very high power microscope you will be not; you will not be able to distinguish it from an ordinary piece of Euclidean space.

We are not interested in general topological manifolds which are those objects that just locally look like Euclidean space, but we are interested in manifolds on which we can do calculus that is where we can integrate and differentiate. So, these will be manifolds that have some notion of smoothness built into it ok.

So, we will restrict ourselves to what are known as C k smooth manifolds that are embedded in R n. So, embedded submanifolds of R n is what we will consider. But since these are the only manifolds that we consider we will just use the term manifold rather than an extremely elongated phrase embedded sub manifolds of R n.

Examples of manifolds include cylinders, spheres, ellipsoids and even somewhat exotic figures like the mobius strip which is a one sided surface. So, since we are dealing only with manifolds that are subsets of R n in this course we do not need to impose any technical condition like separability or para compactness. In fact, seeing manifolds in this special setting will pave the way when you do a further course on general smooth manifolds as to why such conditions are necessary, these conditions are automatically inherited in our case ok.

So, I said before we formalize we need to have a picture of what is it that we want to formalize the pictures you should have in your mind of are that of spheres and ellipsoids and cylinders. So, how do you capture something that looks like a sphere or an ellipsoid or whatever, well we already have one straightforward answer and this is an implicit answer implicit or let me use green for definitions these are provisional definitions.

So, we have an implicit notion of manifold which is a subset S of R n that is locally a hypersurface. So, this is a subset which need not be globally defined by one single function, but near the vicinity of any point of S we should be able to find a function that is whose level hypersurface is going to be exactly this set ok. So, a subset that is locally a hypersurface we have already studied these global hypersurfaces and we have now shown that they behave exactly the way we think a surface should behave they have well defined normal they have well defined tangent spaces and so on ok.

Now, this is an implicit definition simply because at each point the surface will be described by an equation whose 0 set is this piece of the surface, but we have seen that the moment you have a well-defined normal or whatever you immediately have the implicit function theorem that will give you the second definition graph. The graph definition a subset S of R n is a manifold; is a manifold if it is locally the graph it is locally the graph of a smooth function; graph of a smooth function.

So, the first implicit definition deals with equations, the second one deals with graphs and the first definition immediately gives rise to the second definition ok since we have the implicit function theorem. Now, graphs as we have seen in the proof of the lemma that the hyper plane is in fact, an n minus 1 dimensional and so on, immediately gives rise to a parameterisation this was the map capital G which we had defined in that lemma that is the parametric form the parametric definition a subset S of R n is a manifold if it can be locally parameterised it can be locally parameterised meterised by a smooth function.

So, one definition implicit immediately leads to the graph definition immediately leads to the parametric definition. So, our intuitive idea of capturing something that locally looks like a piece of Euclidean space on which we can do calculus has resulted in three provisional definitions, which one do we choose and how do we make these precise.

These are provisional not fully rigorous definitions how do we make this precise thankfully all three definitions give rise to the same notion and you can take any one of them as the definition of a manifold ok. So, let us move on to the definition of a manifold in fact, definitions of a manifold.

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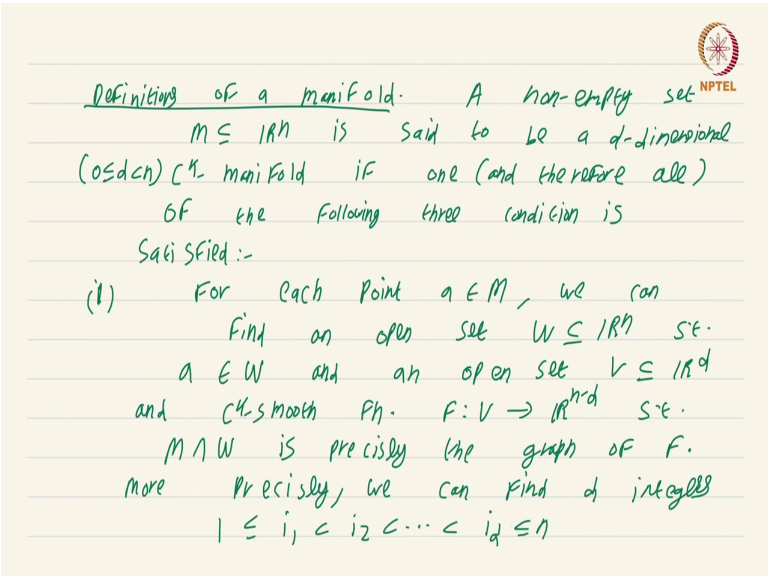

So, we have three definitions of a manifold ok. So, we start with the non-empty set in this course we will not be considering empty set as manifold for the time being a non-empty set M subset of R n is said is said to be a d dimensional C k manifold if one and therefore, all therefore, all of the following three conditions is satisfied is satisfied ok.

So, I must mention d is 0 less than or equal to d is less than n ok, we will talk about n dimensional manifold sitting in R and as part of the definition they are just open sets. So, the first condition is for each point a in M we can express M as a graph near a we have to make that precise and this is going to look somewhat extremely overly complicated notation.

The reason why I am going to put this overly complicated notation is because a graph just means that some variables can be expressed as a function of the other variables I want to make that precise once at least and after that we will generally not be so extremely pedantic.

So, for each point a in M, we can find an open set we can find an open set open set W in R n such that a is in W and a open set an open set V subset of R d and a function and a C k smooth function C k smooth function F from V to R n minus d such that M intersect W is precisely the graph of F. So far so good intuitively clear not too many technicalities now I am going to make precise what is the meaning of M intersect W is the graph of F. More precisely we can find d integers we can find d integers 1 less than or equal to i 1, less than i 2, less than dot dot dot less than i d less than or equal to n. So, you can find d integers i 1 to i d.

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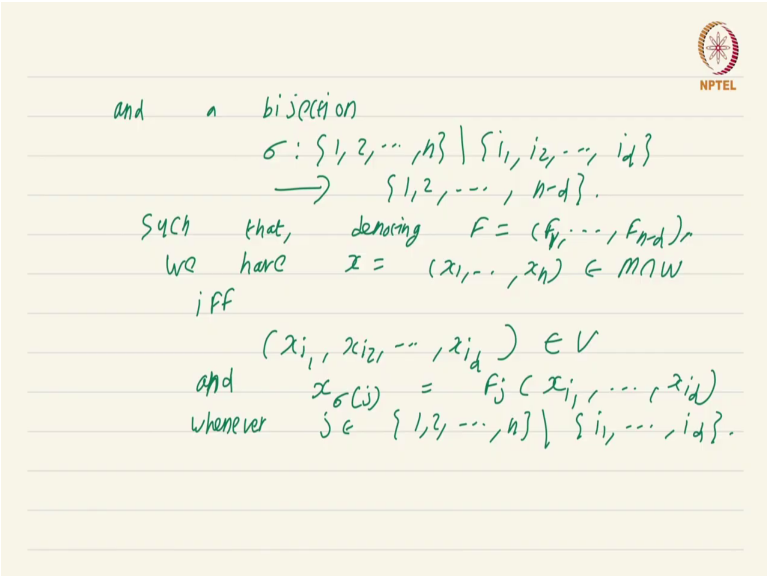

And a bijection; and a bijection sigma from 1 2 dot dot dot n minus these d integers i 1, i 2, dot dot dot i d to 1, 2 dot dot dot n minus d keep this in mind. So, there are exactly d integers i 1 to i d. So, if you remove that from 1 to n you will be left with n minus d integers, you just map these n minus d integers bijectively onto 1 to n minus d. So, the definition is we can find such integers and a bijection such that something happens such that denoting F as F 1 to F 1 comma dot dot F n minus d remember F was a map from an open set V in R d to R n minus d.

So, there will be n minus d coordinate functions such that denoting this we have x equal to x 1 dot dot dot x n is an element of M intersection W if and only if; if and only if x i 1, x i 2 comma dot dot dot x i d is an element of V and x sigma of j is nothing but F j of x i 1 comma dot dot dot x i d whenever j is in 1, 2 dot dot dot n minus i 1 dot dot dot i d ok. This looks to be a impenetrable fortress of a definition, but its just making precise what it means for a set to be a graph. Well, what is it saying?

In the simplest situation what we will have is the first d variables would be the independent variables and the final n minus d variables will be the dependent variables and this function this set M intersection W would be written as a graph of the first d variables, but life is no so kind the dependence could be very complicated it could happen that the dependence is on the first, third, fifth, seventh variables and so on.

So, what we do is this i 1 to i d gathers together all the independent variables ok i 1 to i d gathers all the independent variables. So, the rest of the variables is supposed to be the variables that are going to be written as a function of these d variables ok. Now, the function F is going to be the function that writes it as a graph.

So, the function we are denoting by F 1 to F n minus d, this sigma map just puts this F 1 to F n minus d in the correct slots to make it a graph and we have made that precise by saying that a point x 1 to x n is there in M intersection W first of all the independent variables should belong to V. So, you look at all the i 1, i 2 th i 2 dot dot dot i dth coordinate of this x 1 to x n and you check that it is there in V ok.

So, the independent variables are in fact in V then you have to check that the dependent variables are in fact in this I mean the dependent variables in fact match with the values of the function F and that you do by checking that x sigma j is nothing but F j of x i 1 to x i d ok. So, indeed it is happening that the dependent variables are arising as the functional values of this function F therefore, this M intersection W is indeed a graph.

So, we have made precise what it means for M intersection W to be a graph and I am admittedly this is a very ugly thing, but it is somewhat elegant if you think about the way we have formalized ok. So, this is one definition; this is one definition.

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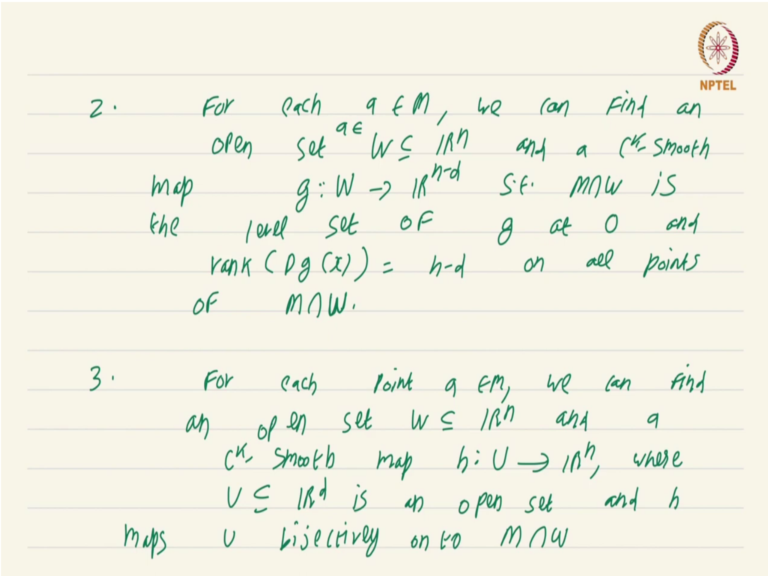

The second definition is the second definition is the one where you have it directly as the level set of a function. So, for each a in M we can find; we can find an open set W subset of R n and a C k smooth map; C k smooth map let us say g from W to R. So, this is going to be a somewhat more general thing g from W to R n minus d. So, far we considered only hypersurfaces that will just give you n minus 1 dimensional manifolds now we want to consider level sets of functions whose co domain is not just R, but R n minus d ok.

So, we have a map g from W to R n minus d such that M intersection W is the level set; is the level set; is the level set of course, I must mention a is in W a is in W is the level set of; is the level set of g at 0 and rank of D g x rank of D g x is n minus d on all points; on all points of M intersect W ok. So, usually what happens is we consider only co domain R. So, things become rather simple in that situation.

Here we have co domain n minus d in that scenario we had d to be n minus 1 and n minus d to be 1. So, just the gradient non vanishing is same as saying rank D g x is equal to n minus d which in that case was 1. Here we have to specify that rank D g x is n minus d; that means, it is a full rank matrix, we will see a little bit more about what this definition is trying to say in a moment, let me just finish this and give the third definition.

For each point a in M we can find; we can find an open set we can find an open set W subset of R n and a C k smooth map; and a C k smooth map h from U to R n, where U subset of R d is an open set; is an open set and h maps h maps U bijectively bijectively on to M intersect W.

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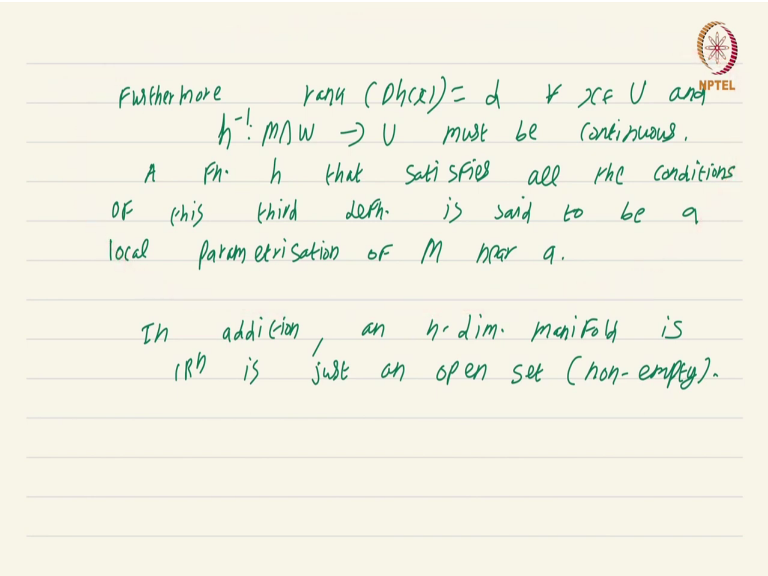

Furthermore we want the rank of D h x to be equal to d this time not n minus d for all x in this set U and h inverse from M intersection W to U must be continuous; must be continuous ok. So, this is the parametric form of the definition ok. Now, let me just mention a function h that satisfies all the conditions of this part conditions of this part this third definition is said to be is said to be a local parameterisation local parameterisation of M near a ok it parameterises a portion of the manifold around the point a.

So, before moving on to the comments about this definition let me just give what this n dimensional manifold sitting in R n is going to be in addition. So, this is apart from these three conditions in addition an n dimensional manifold; n dimensional manifold in R n is just an open set; is just an open set non empty ok. We had taken d to be less than n in the above three criterion in the scenario where you want to study an n dimensional manifold in R n it is just an open set.

Now some comments about this definition many of which will be trivial notice that the smoothness class when you say C k manifold that is determined; that is determined by the smoothness of the map under consideration in each one of these definitions. So, here we had F to be a C k smooth function in the first definition, in the second one we had g to be a C k smooth function and in the third definition we had h to be a C k smooth function ok.

So, that is the first comment typically for this course just C 1 is enough we will usually be considering just C 1 manifolds, the most important class of manifolds are C infinity manifolds that is these functions h, g and F they are all C infinity smooth. So, in this k k is allowed to be infinity that is another comment, third comment is any C k manifold is automatically a C k minus 1 manifold for obvious reasons that is the third comment.

Fourth comment is this neighborhood W this open set W that we are considering for notational simplicity we have used the same W for all three definitions, but they need not be the same; obviously, each definition could give you a different W for which the conclusion of this definition is true, it need not be the case that the W that you choose here will work for the W that you choose here that might not happen.

So, be careful this is just to make sure that we do not get overburdened with notation that we have reused the same letter W ok. So, now, more specific comments about the definition the first condition as I have already mentioned is just an overly pedantic way of saying that locally the set M looks like the graph of a C k smooth function. Now, loosely speaking a d dimensional manifold is an object on which we have d degrees of freedom. So, this d degrees of freedom is determined by the d independent variables ok. So, the first definition sort of models that.

The second definition also models that if you think about it, each additional coordinate in the co domain adds on one constraint for this set right. So, it this is a thing that satisfies the g of x equal to 0 that is precisely the points of this manifold. So, you can think of each additional coordinate in the co domain as adding a constraint and reducing the degree of freedom by 1 ok. So, if you have n minus d in equations then the degrees of freedom you have are d exactly. So, this is another way of looking at the second definition.

Now third definition I will have more to say about it in the examples video which will which is the next video in fact. So, this third condition looks extremely complicated compared to the second condition it is modeled on the second condition the graph; obviously, parameterizes this, but why have we kept such a complicated definition for when you want to consider parameterisations.

Well, the reason is as follows the reason is that there are certain things that are automatic for graphs for instance this x comma F of x that is how the graph x comma g of x in this situation that is the graph that will automatically be full rank right. So, you do not need to put that condition and the inverse map will automatically be continuous because the inverse of the graph is just a projection map if you think about it.

So, certain things are automatic in the graph so you have to make that explicit when you look at the parameterization, the conditions will become very clear when we talk about examples and you will see several exercises where what will go wrong if you drop one of these conditions, you will have figures which are; obviously, not going to look locally Euclidean become manifolds ok.

Now, final comment sometimes we have a map h from an open set U to the manifold M that need not satisfy all these conditions ok we have a lot of conditions this map h mean need not satisfy all of this. So, h nevertheless sort of parameterises M near this point a, but it does not parameterise it nicely in the sense of this definition 3, but we will often have to consider such maps in fact, what will happen is we will have a global map to M which a smooth map let us say from h from an open set U to M that may not be full rank at all points and so on.

But it might be possible to manufacture from this function h a local parameterisation in the sense of this definition. So, we will sometimes say that a map that takes an open set U and maps it to a portion of M even if it does not satisfy all these conditions we might still continue to call it a parameterisation. So, that is just an abuse of terminology it is just for convenience I do not want to invent a new term for that specific situation that arises so often.

So, I am just going to call that also a parameterization, but we will make clear when we want to say something is a local parameterization of M near a we mean something that satisfies all these conditions ok.

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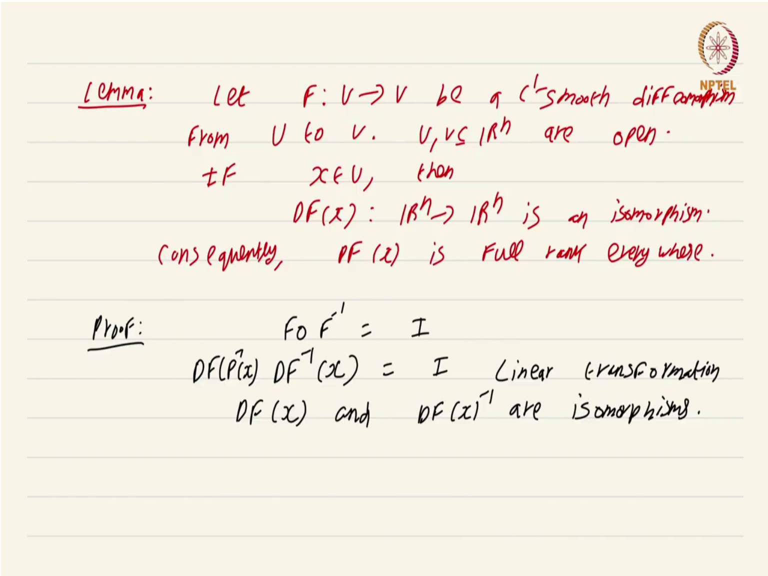

So, let us begin with the proof of the equivalence the proof is somewhat long, but not so difficult let us just illustrate, what the rank where the rank comes into the picture with the small lemma with a small lemma. Let F from U to V be a C 1 smooth C 1 smooth diffeomorphism from U to V this just means that the inverse map is also a C 1 map and F is bijective ok, of course, U and V subset of R n are open.

Now if x is a point of U then D F x D F x from R n to R n is an isomorphism; is an isomorphism. Consequently, D F x is full rank everywhere. The proof of this is a triviality the proof of this is a complete triviality so, we have F composed with F inverse is identity.

So, the chain rule says D F inverse at a point x and so, to be ultra precise D F at F inverse x multiplied by F inverse x is nothing but the identity linear transform identity matrix identity linear transformation not matrix identity linear transformation and that is it this completes the proof basic linear algebra will tell you that D F x and D F x inverse are isomorphisms. So, I am not going to elaborate anymore this is a rather simple thing ok.

Now, this is just to illustrate about full rank we are not really going to use this lemma in this proof let us begin with the proof, we will first show no let us solve the easy cases.

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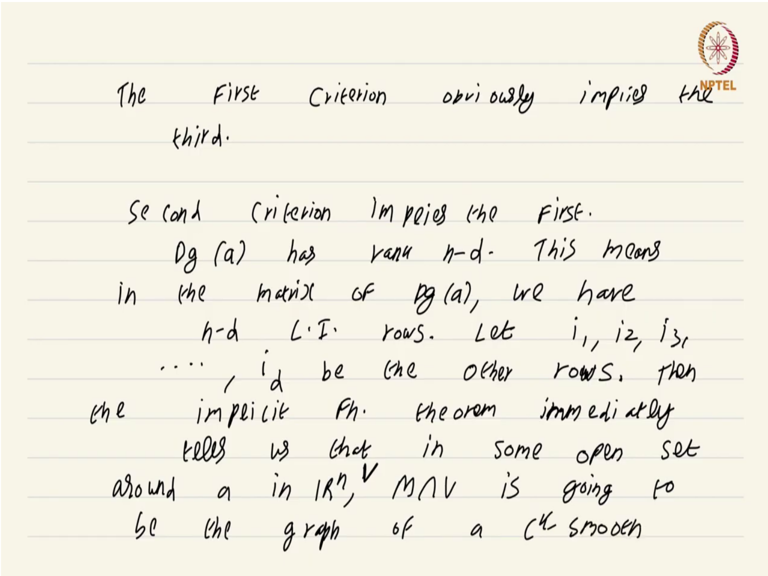

The third the first criterion obviously, implies the third; implies the third I had made a comment about this a graph is automatically going to be a parameterisation that satisfies all the conditions of part 3. The inverse being continuous is just because the inverse is the projection it being full rank is because it is a graph it is it this is a very straightforward thing and left to you ok.

Now we are going to prove that second criterion implies the first. So, we have to show that if it is locally the level set it is going to be locally a graph ok. Now, the assumption is D g at a has rank n minus k n minus d sorry D g at a as rank n minus d. Now this means in the matrix; in the matrix of D g a we have n minus d linearly independent rows this is just the rank being n minus d translated into matrix form so, it will have n minus d linearly independent rows ok.

Let i 1, i 2, i 3 comma dot dot dot i d be the other rows ok, consider those to be the other rows then the implicit function theorem says. Then the implicit function theorem implicit function theorem immediately tells us; immediately tells us that in some neighborhood some open set around a in R n, M. So, let us call this we will use a different thing let us say v there is this open set let us call it V, M intersect V is going to be; is going to be the graph the graph of a C k smooth function; C k smooth function of x i 1, x i 2, dot dot dot x i d ok.

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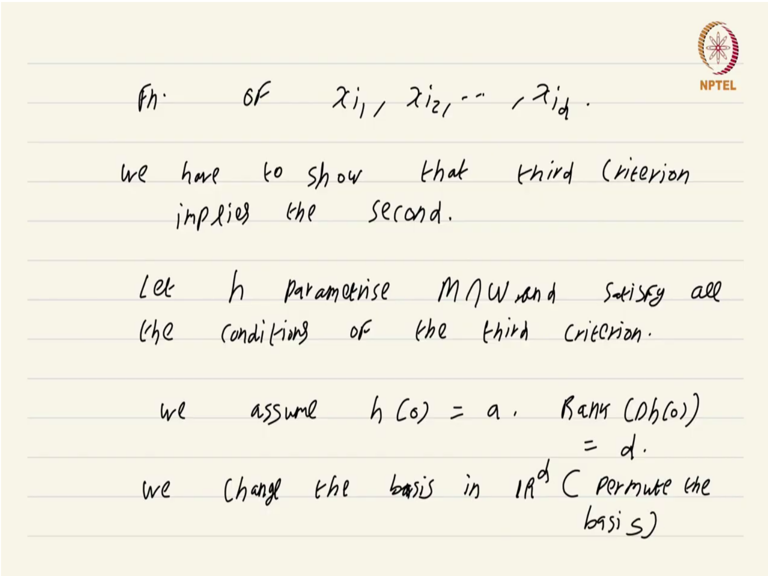

So, these x i 1 to x i d are going to be the independent variables and this function is going to determine the n minus d the rest of the variables and this set M intersection V is going to be the graph. This is just a straightforward application of the implicit function theorem it is going to be left to you ok, I must mention when you take these other rows you take it in order ok.

You take these other rows in order because when we said graph we had defined i 1 should be the first i 2 should be the second they should be in order. So, you just take these rows in order the other rows in order and then you just apply the implicit function theorem just check the hypothesis this is just a matter of unwinding the statement of the implicit function and the situation we are in it will immediately follow ok.

So, I am going to leave the details finishing this part that the second criterion in fact, implies the first to you the rest of the proof is rather easy ok. Now, we have to show the toughest part we have to show; we have to show that third criterion implies the second implies the second that is if you can locally parameterise a piece of M then you can write a portion of M also has a level set of a function that has all these properties that we need from criterion 2 ok.

So, let h parametrise M intersection W ok and satisfy and satisfy all the conditions all the conditions of the third criterion that is it is a local parameterisation near the point a of the third criterion ok. We must somehow manufacture a function whose level hypersurface is going to coincide with this m at least locally ok.

Now we know that; we know that D h first of all ok first of all we assume h of 0 is equal to a ok now we also know that rank D h 0 is equal to d this is part of the hypothesis of what it means for h to be a local parameterisation. So, what we do is we change the basis; we change the basis essentially this is just going to be a permutation we change the basis in we change the basis in R d.

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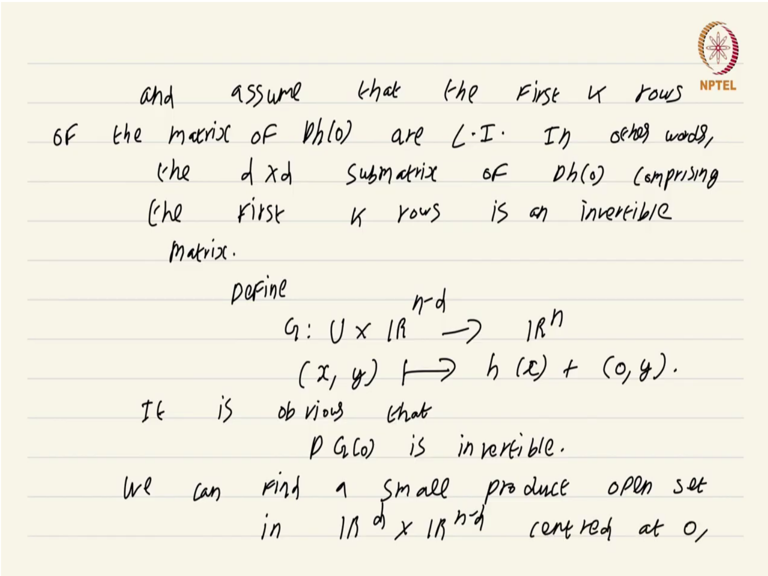

So, this is just by a permutation permute the basis we permute the basis and assume that the first k rows of D h 0 are linearly independent ok the matrix to be 100 percent precise first k rows of the matrix of the matrix of D h 0 are linearly independent. We can do this there will be k rows that are linearly independent you just permute the coordinates to make those k rows the first k rows of the matrix of D h 0 ok.

So, in other words; in other words the d cross d sub matrix of d h 0 comprising the first k rows first k rows is an invertible matrix again this is just saying the same thing in a different way invertible matrix ok. Now this is this remark that the first k rows is an invertible matrix is going to become very important in a moment.

Define G from U recall U was the domain of h U cross R n minus d to R n this is a map that takes the point x comma y x is in u and y is an R minus d and maps it to h of x which will be an element of R n plus 0 y ok we augment d zeros and then add y I mean then put y in the second coordinate. So, this will also be this second part will also be in R n and you just add these two ok.

So, why are we doing this? Well we want to somehow apply the inverse function theorem the problem means dimensions do not match. So, we are going to make the dimensions match by tagging on new variables n minus d of them, but we want to may do it in such a way that there is invertability. So, that is why we are tagging on 0 y and it will become very clear why we assume that the d cross d sub matrix of the first k rows are exactly the ones that are they are going to be linearly independent because it will now be very obvious it is obvious it is obvious that that D G of 0 is invertible.

Please check that please check that what will happen is uh there will be a k cross k sub matrix not k cross k d cross d sub matrix which is going to be invertible contributed by h then the remaining one is sort of like identity. So, I am going to be a bit vague in this. So, essentially it will be a block matrix you can check that easily that the matrix of this D G of 0 is going to be invertible just write down the matrix and you it will be very clear to you ok.

Now, that we have shown that D G of 0 is invertible or rather you are going to show the D G of 0 is invertible we can apply the inverse function theorem. What is the inverse function theorem, say well we can find; we can find a small product open set in R d cross R n minus d centered at 0 say V 1 cross V 2 V 1 cross V 2.

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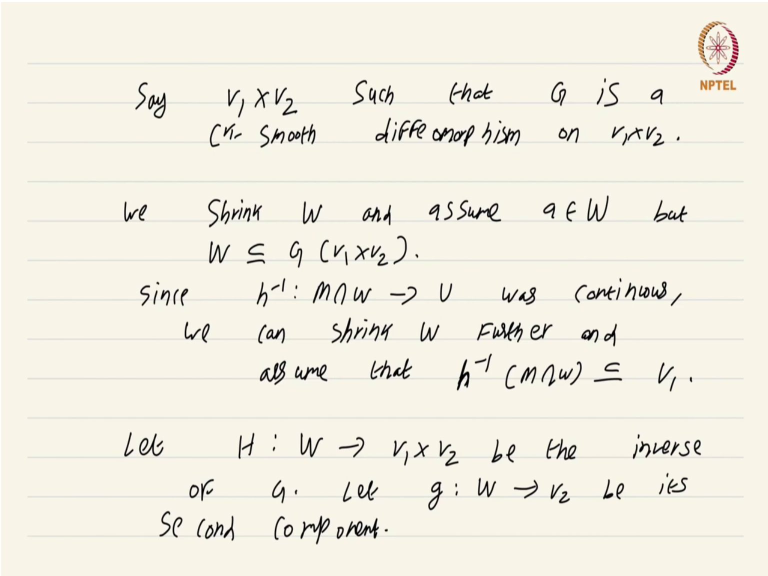

So, V 1 is an open subset of Rd V 2 is an open subset of R n minus d we can write we can find a small open product open set V 1 cross V 2 such that G is a C k smooth diffeomorphism on V 1 cross V 2 ok. So, that means, G from V 1 cross V 2 on to G of V 1 cross V 2 and that inverse is also going to be a C k smooth diffeom C k smooth map that is what it means ok. We shrink W; we shrink W and assume that a continues to be in W, but W is a subset of G of V 1 cross V 2 this can be done we just shrink this W so that this W lies entirely in the image of G of V 1 cross V 2 ok.

So, far we have just used the rank condition on the map F on the map h to get this inverse get this diffeomorphism G using the inverse function theorem now what we are going to do is we need to invoke the fact that h is h inverse is also continuous. Since h inverse from M intersection W to U was in was continuous we can shrink; we can shrink W further and assume that h inverse of M intersect W is fully contained in V 1 ok.

We can do this as well because h is h inverse is continuous on M intersection W. So, we just shrink W even further so that h inverse of M intersection W is fully contained in V 1 this is just characterizing the epsilon delta definition for the continuity of h inverse. So, this open set V 1 you choose an appropriate epsilon so that then 0 neighborhood of 0 neighborhood an open ball centered at 0 fully contained in V 1 you choose an appropriate delta that will tell you how much to shrink W ok.

Now, what we are going to do is we are going to see what we can do with this map G we know that this map G is a C k smooth diffeomorphism. So, what we do is let capital H let capital H from W to V 1 cross V 2 be the inverse of G. We know that H is also a C k smooth map. Let g from W to V 2 be it is second component be it is second component that is H I am writing it as two components one that maps into V 1, one that maps into V 2 I am just going to consider g to be the second component the component that maps to g 2 ok.

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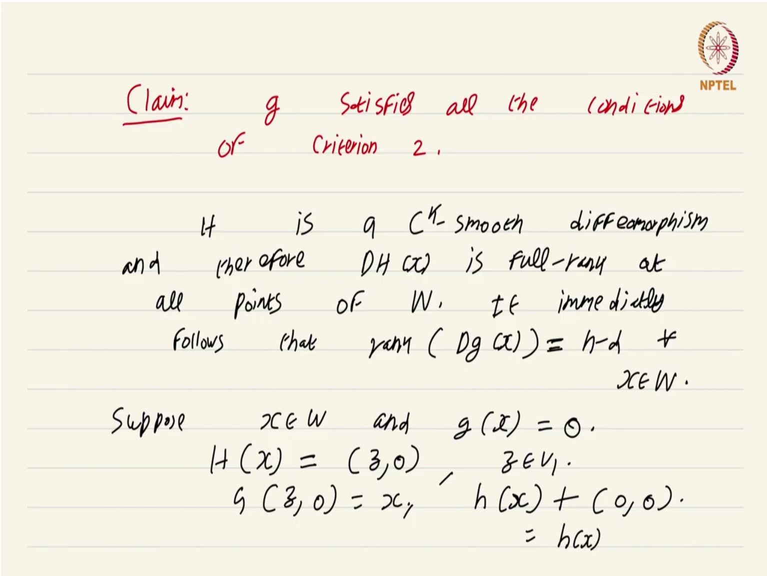

Now, this we are at the finishing stages of this proof as the notation suggests this g does the job g satisfies all the conditions all the conditions of criterion 2 criterion 2 ok. How do we show this, well we know that D H of x is a C k smooth not D H of H is a C k smooth C k smooth diffeomorphism and therefore, this is the only place I am going to invoke the previous lemma D H x is full rank; is full rank at all points; at all points of W points of W ok.

Now, since D H x is full rank at all points of W it immediately follows; it immediately follows that rank of D g of x is exactly equal to n minus d for all x in W this is immediate ok. So, we have to show now that the level set of this function g at 0 is precisely M intersect W.

So, suppose x is a point in W and g of x is equal to 0 we have to now show that x is actually a point in M intersection W M is it is also a point in M, but what is g of x equal to 0 saying which just says that H of x is going to be of the form z comma 0 ok where z is in V 1.

So, in other words shifting this back G of z 0 is going to be in is not going to be in is exactly equal to z is exactly equal to x is exactly equal to x, but this is just h of x; h of x plus 0 0 right that is the way this map g was defined it is just h of x plus 0 0 which is just equal to h of x ok.

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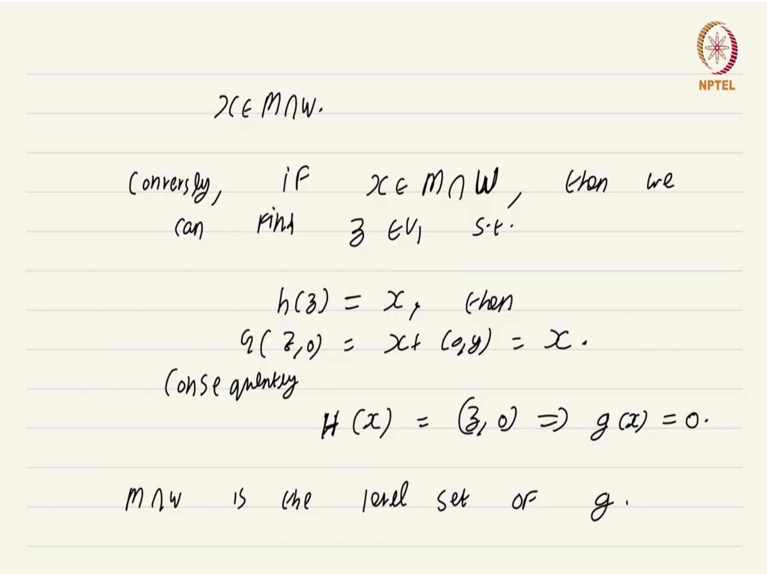

But h maps on to M intersection W. So, this means that x is in M intersection W. So, that was just showing that if you have a point x in M intersection W then G of x equal to 0. Conversely, if x is in M intersection W then we can find; we can find z in V 1 such that h of z; h of z is equal to x right, then h of not h of G of G of z comma 0 is x plus 0 y is equal to x.

Consequently, capital H of this point x has got to be z comma 0 which just means g of x is 0 ok. So, what we have shown is if you start with the point x such that g of x is 0 then this point x has to be in M intersection W if you start with the point x in M intersection W then g of x equal to 0, consequently M intersection W is the level set; is the level set of g and this concludes the equivalence of the three definitions of a manifold that we have given.

Now, just one remark I want to make I had defined a n dimensional manifold in R n as just any non empty open set in R n now there is a way to write out this three equivalent conditions that does not depend on d being less than n minus 1 ok. So, for instance you can write the open set U in R n as a graph of a function defined on R n, but whose co domain is R 0 which is just the set consisting of the single vector 0.

Now, it is a bit inelegant in my opinion to do it that way. So, I am not doing that I am just specifying d equal to n k separately I am doing that separately because I do not want to treat graphs taking values in the set with 0 and all that it is a bit complicated, but nevertheless you should know that it can be done.

So, this concludes the proof this is a course on Real Analysis and you have just watched the video on The Definition of a Manifold.