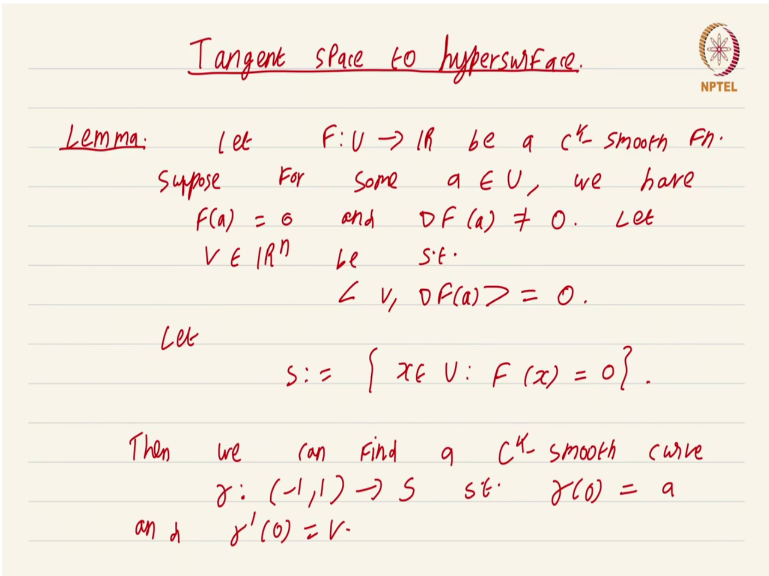
**Real Analysis - II**

**Prof. Jaikrishnan J**
**Department of Mathematics**
**Indian Institute of Technology, Palakkad**

**Lecture - 19.1**
**Tangent Space to a Hypersurface**

(Refer Slide Time: 00:22)




In this video we are going to use our new tool the implicit function theorem to justify one of the definitions that we had given the definition of the Tangent Space to a Hypersurface. Recall that a hypersurface was nothing but a level set of a C 1 smooth function such that the gradient of that function does not vanish at any point on the level set. We had also defined the tangent space to be the collection of all velocity vectors of curves that lie entirely in that hypersurface.

The only issue is that it is not even obvious that there is one curve that lies entirely in the hypersurface other than a constant curve of course. So, now, we are going to fix these issues and make everything rigorous, I will prove a slightly more general lemma that does the job for us. Lemma let F from U to R be a C k smooth function, suppose for some point a in this open set U we have F of a is equal to 0 and the gradient of F at a is not 0 ok.

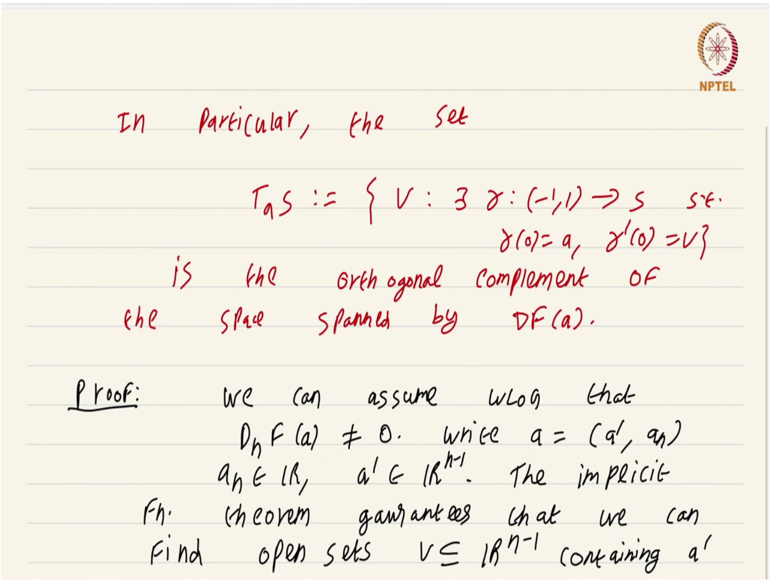
Let V in R n be such that the inner product V comma gradient of F at a is 0, in other words this vector V is normal to the gradient of F of a, recall that the gradient of F of a was suppose to be the normal to the hypersurface. So, this vector V is normal to the normal in other words V is suppose to be a vector in the tangent space. Let S by definition be the set of all points U set of all points x in U such that F of x equal to 0.

Note in the slightly more general setting S need not be a level hypersurface, I am just requiring that the gradient of F at a not vanish. So, essentially I am taking the condition in the definition of hypersurface and focusing my attention near a point a so that is why I said this lemma slightly more general than what we need to prove. But, since this is a local thing existence of a normal and existence of a tangent hyper plane they are all sort of local notions it is better to prove it in this setting to have clear understanding as to what is relevant and what is not relevant ok.

So, this is the setup as follows we have this function F we are considering the level set of this function which need not be a hypersurface, but at least at the point a the normal is well defined the conclusion is as follows. Then we can find we can find a C k smooth curve gamma from minus 1, 1 to S such that gamma of 0 is equal to a and gamma prime at 0 is the vector V ok.

So, what this shows is that given any vector that is normal to the normal then we can find a curve that passes through this hypersurface that in fact, lies entirely in this hypersurface such that the velocity vector of this curve is nothing but the pre specified vector V ok.

(Refer Slide Time: 04:47)




So, we have a second part of this conclusion in particular the set T a S which is exactly defined as in the case of hypersurfaces the set T a S such that V such that there exists gamma from minus 1, 1 to S not R n S such that gamma of 0 equal to a gamma prime of 0 equal to V is the orthogonal complement of the space spanned by gradient of F at a ok.

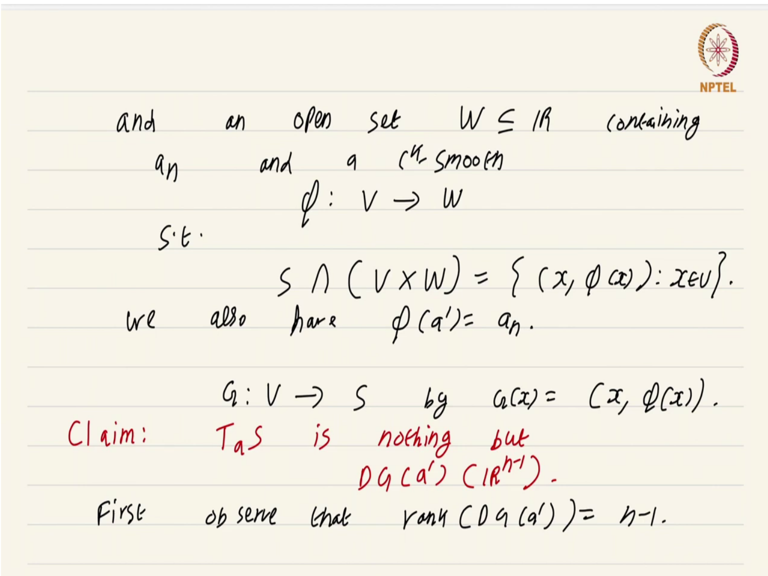
So, we have got several conclusions one is that any vector that is normal to the normal will be a velocity vector of a curve passing through the surface and using that we will show that the collection of all such velocity vectors is exactly the orthogonal complement of the space spanned by gradient of F of a ok.

So, one of the reasons why I wanted to prove this rather than just directly moving on to the theory of manifolds, where we will study the tangent space again in more general situation is that this particular lemma though very simple and straightforward illustrates the implicit function theorem rather visually. And if you understand this lemma what is about to follow the general notion of manifolds and the notion of tangent space in general will become quite easy ok.

So, since the gradient is not 0 we can assume without loss of generality we can assume without loss of generality that the nth derivative D n of F at a is not 0 this essentially just means renumber the coordinates. So, that the last vector in the last sorry the last number in the gradient is not 0 we can always do this, write this point a as a prime comma a n where of course, a n is a real number and a prime is in R n minus 1 ok.

Now, because the nth derivative is not 0 the implicit function theorem immediately gives us the implicit function theorem guarantees that we can find we can find open sets V subset of R n minus 1 containing the point a prime containing a prime.

(Refer Slide Time: 08:12)



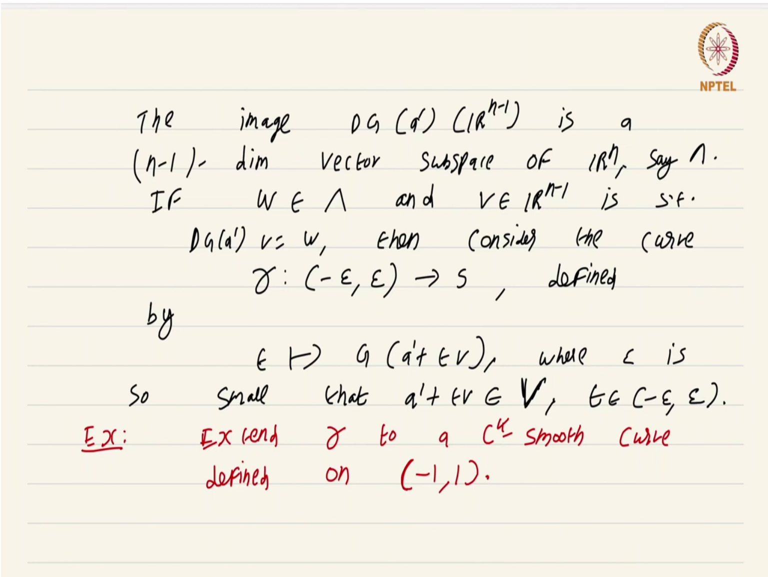
And an open set W in R containing a n containing a n and a C k smooth map that is essentially going to say that this piece of the hypersurface actually it is not really a hypersurface this piece of S near a is going to be a graph. So, we can find a C k smooth function phi let us say from V to W such that S intersection V cross W that is just the portion of S in this product neighborhood is going to be nothing but the graph x comma phi of x such that x comes from U ok.

So, this immediately follows from the implicit function theorem we also have we also have phi of a prime is a n ok. So, what we have essentially done is that we have expressed a portion of S near this point a as the graph of a C k smooth function ok.

And another interpretation is phi this function phi is sort of parameterizing the surface S near the point a ok. Now, what we are going to do is define this graph map essentially G from V to S by G of x is nothing but x comma phi of x ok. Now, what we are going to show is T a S so, rather I can write this as a claim what we have defined as T a S is nothing but D G at the point a or rather yeah D G at the point a prime of R n minus 1 ok. This D G of a prime is supposed to be the derivative of this map G and we are going to claim that the tangent space T a S that we have defined is nothing but the image of this map at the point a prime the entire image ok.

So, to prove this first observe that rank of this D G a prime is; obviously, n minus 1 it is; obviously, n minus 1 it is a full rank matrix simply because the first coordinate is just G of x equal to x comma phi of x. So, this will be immediate by looking at the matrix of this map D G a prime from there it will be really clear ok.

(Refer Slide Time: 11:43)




Now, because the rank is n minus 1 the image that we are interested in the image D G a prime of R n minus 1 is a n minus 1 dimensional vector subspace of R n excellent. So, we are getting somewhere we have got that this vector subspace is going to be n minus 1 dimensional ok. Now, what we have to show is that this vector subspace is nothing but T a S that is the ultimate goal so call this vector subspace capital lambda let us call this vector subspace capital lambda.

What I am going to show is that if W is a vector in this lambda and V in R n minus 1 is such that is such that D G a prime V equal to W then I am going to manufacture I am going to manufacture a curve that goes through the surface S and has velocity vector at 0 exactly this vector W exactly this vector W. How am I going to do that? Then consider the curve gamma from minus epsilon, epsilon minus epsilon, epsilon to S defined by t as you can guess maps to

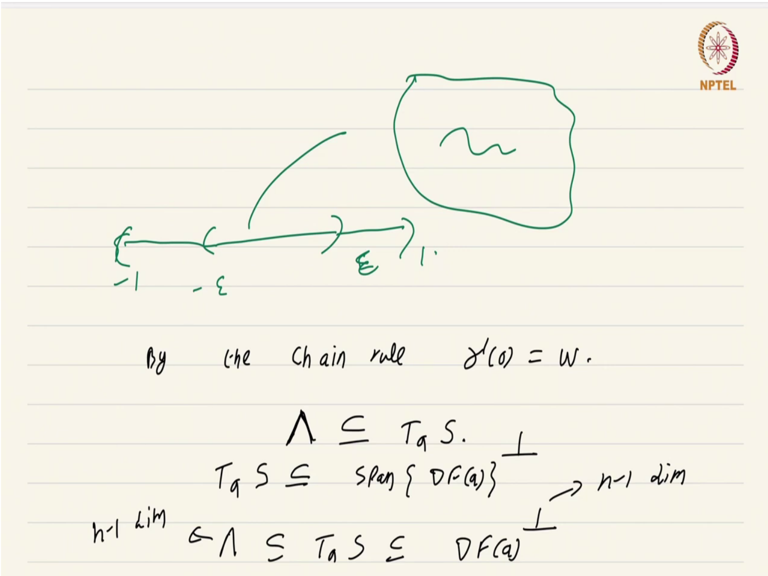
G of a plus t v ok. So, what I have essentially done is I know that this map G parameterizes S near this point a, I know that D G a prime of V equal to W.

So, what I am just going to do is I am just going to consider the curve that is in the direction V that is just a straight line in the direction V lying in the domain V. And then I am just going to act G on that and this will make it into a curve that lies in the surface S and of course, because of the way things have been defined one second there is a slight error here this should be a prime this should be a prime because of the fact that G of a prime is nothing but a prime comma a n which is nothing but a this curve does indeed pass through the point a. And the way things have been set up the derivative will be immediately be seen to a ok.

Now, this epsilon is so small where epsilon is so small is so small that a prime plus t v is entirely in this neighborhood V in this open set V ok, t coming from minus epsilon epsilon ok. Now, in the definition of the tangent space I had given for concreteness sake I had restricted myself only to curves whose domain is the specific domain minus 1, 1. Now, this curves domain is certainly not minus 1, 1 unless this set V happens to be somewhat big it is just minus epsilon epsilon. So, I am going to leave it as an exercise for you exercise.

So, extend gamma to a C k smooth curve to a C k smooth curve defined on minus 1, 1 defined on minus 1, 1 ok. So, what is happening is this curve gamma now is just defined on minus epsilon epsilon this is actually irrelevant actually what the domain of this curve is as long as gamma of 0 is equal to this point a, it really does not matter what the domains of the curves are I am asking you to make that precise ok. Show that gamma can be made into a C k smooth curve defined on minus 1, 1.

(Refer Slide Time: 16:17)




So, you have gamma which is defined on a portion you have gamma which is now defined on a portion this is going to. So, let us say this is the surface you have the image of this what I am saying is you can define it you can extend it to minus 1, 1 also minus 1, 1 in such a way that the curve continues to be a smooth curve ok. So, this is a somewhat little bit of thinking needs to be done to solve this, but it is a very interesting exercise for you to grapple with ok.

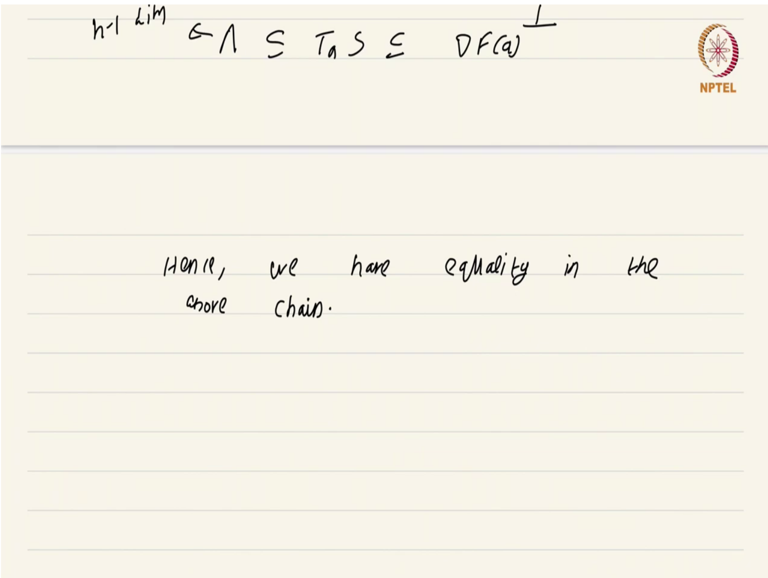
So, we have extended this curve to a curve on minus 1, 1. Now, by the chain rule by the chain rule gamma prime of 0 is nothing but this vector W that we wanted ok. Now, this what we have essentially shown from this consideration is the fact that this vector space lambda that we have this vector space lambda that we have remember that was the image of D G a we picked W from that image.

So, this looks like a capital W. So, let me just make it small w ok. So, we picked this vector w from this lambda and what we have shown is that there is a velocity there is a curve whose velocity vector is exactly w so; that means, that this gamma sorry lambda capital lambda is nothing but a subset of T a S ok. Now, the claim is that lambda is actually equal to T a S which will prove that T a S is also n minus 1 dimensional.

But, we have earlier shown we have earlier shown that T a S is the subset of this span of gradient of F at a orthocomplement ok. We have already shown this of course, there we showed it for a hypersurface, but the proof is entirely local it really does not use the fact that S is globally a hypersurface only the behavior of the normal at the point a really matters for that proof ok.

Putting all this together we have this chain lambda is subset of T a S is subset of I will just be a little bit loose with notation and just call this orthocomplement gradient of F of a complement. So, we have this chain and immediately we see that this is an n minus 1 dimensional subspace. So, this is n minus 1 dimension this is n minus 1 dimension because it is the complement of a vector in R n. So; that means, all three have to be equal.

(Refer Slide Time: 19:18)




So, hence we have equality above we have equality in the above chain ok. So, this concludes the proof that the tangent space is going to be n minus 1 dimensional it is also going to be exactly the complement of the normal and it is also going to be the image of this derivative of this graph mapping.

So, this is an excellent starting point for the theory of manifolds which we are going to begin in full earnest pretty soon ok. So, our picture of what a hypersurface is; is now complete. A hypersurface a level hypersurface is nothing but the level set of a function that has a well defined normal at every point. So, a non-zero normal at all points and now it has a well defined n minus 1 dimensional tangent space that is also obtained as the velocity vectors of curves passing through the points of the surface.

So, our intuitive picture of a hypersurface is now complete we have now made the definition of hypersurface coincide with our intuition correctly. So, this lemma as I keep repeating is very important. So, please understand it thoroughly much of what follow soon will become very easy if you understand this lemma. This is a course on Real Analysis and you have just watched the video on Tangent Space to a Hypersurface.