Real Analysis II
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Lecture - 18.1
Diffeomorphisms and Local Diffeomorphisms

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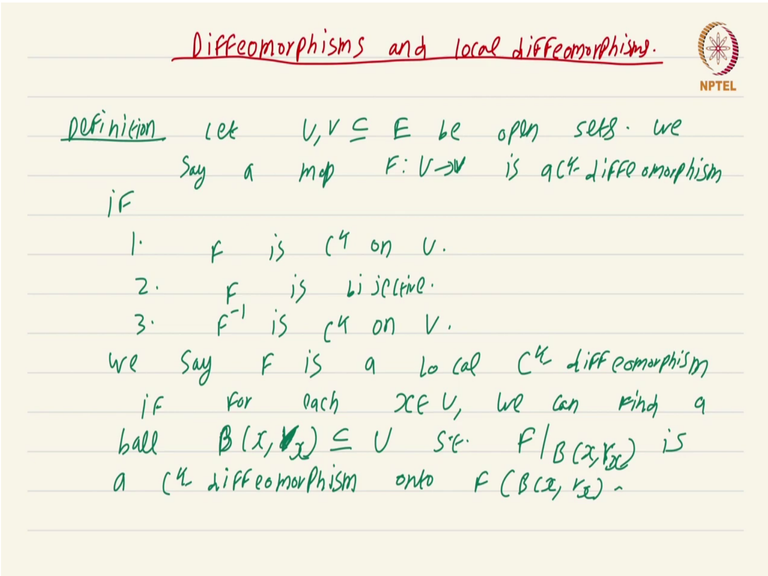

What are the purpose of sophisticated terminology such as Diffeomorphisms. Is it to intimidate the listener, bully the listener into believing that you are very smart no language exists to serve as a short cut to aid thinking? We are now going to define these terms diffeomorphisms and local diffeomorphisms, this will enable us to talk about situations quite freely and in a concise manner.

This these definitions are motivated by the conclusion of the inverse function theorem, that says that if a C k smooth mapping, that derivative of such a mapping, if it is non singular then locally the map is invertible and the inverse is differentiable.

So, we are going to introduce the notion of diffeomorphisms, which is the global version of the conclusion of inverse function theorem and local diffeomorphisms, which is the local version of the same. So, you can concisely describe the inverse function theorem as saying that if the derivative map is isomorphic, then the map is a local diffeomorphism.

So, it sounds fancy, but it serves as a nice shortcut an easy way to remember plus it simplifies our thinking. So, let us make the definition of diffeomorphism and local diffeomorphism definition. Let U comma V, be sub sets of E be open sets, we say a map F from U to V is a diffeomorphism is a diffeomorphism, or rather C k diffeomorphism, if first of all F is C k on U, second F is bi jective, and 3 F inverse is C k on V.

So, it is this notion of a C k diffeomorphism is clearly just the global version of the conclusion of the inverse function theorem, you say that 2 open sets, U and V are diffeomorphic, if you can find a C k smooth diffeomorphism, a C k smooth map, whose inverse is also C k smooth. So, we will generalize this notion of diffeomorphisms to more general sets than open sets quite soon after we introduce the notion of many folds.

As, I said the conclusion of the inverse function theorem is local you do not get a global conclusion I have asked you to come up with a counter example. So, let us make the local version of this definition. We say F is a local C k diffeomorphism; is a local C k diffeomorphism, if for each x in U, we can find; we can find a ball B x, r and this r really is not independent of x. So, let me just call it D r x a ball which is contained in U. Such, that S restricted to B x r x is a C k diffeomorphism is a C k diffeomorphism onto F of B x, r x ok.

So, what this is saying is we say that the map F is a local C k diffeomorphism, if for each point we can find some small ball on which F is L diffeomorphism globally. So, near any point it is going to be a local diffeomorphism, that is the conclusion of the inverse function theorem. So, let us just summarize the conclusion of the inverse function them with this new found terminology.

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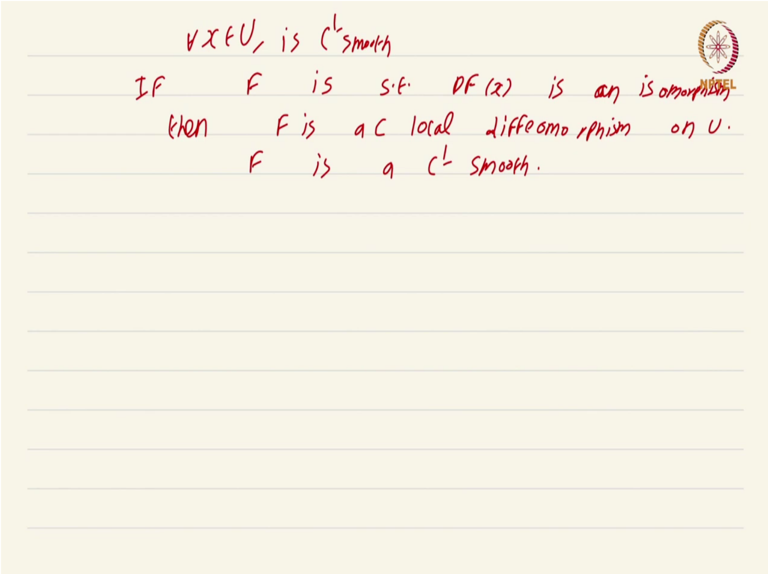

If or let me put it in a red color, if F is such that D F x is an isomorphism of vector spaces of course, then F is a local diffeomorphism on U. So, if for all x in U, for all x in U, F is such that D F x is an isomorphism, then F is a local diffeomorphism on U.

Again please think of a counter example to the conclusion of the inverse function theorem being a global conclusion, it is not true that even if D F x is an isomorphism at all points of U that is not at all true that F is going to be a global diffeomorphism ok. So, I kept saying that, I mean the definitions that I have made; now I kept saying that we have this conclusion.

But, if you recall we just prove the case in the inverse function theorem, we just prove the case when F is a C 1 map right. We did not really talk about the C k smooth case; we did not really talk about the C k smooth case. So, far actually what I have written down in red is not technically correct, what I can really say here is that if F is C 1 smooth and if D F x is an isomorphism, then F is a C 1 or local C 1 diffeomorphism that is all, we can say.

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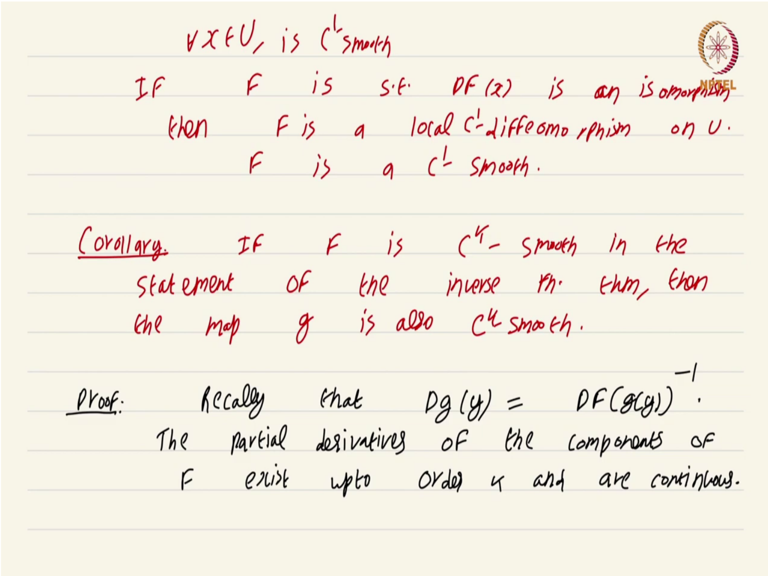

We really do not have the C k conclusion as of it though I said that we have it that is because I am going to prove it right away. This is a corollary of the inverse function theorem it is not that difficult to prove. Corollary, if F is C k smooth in the statement, in the statement of the inverse function theorem.

So, the statement exactly the same statement except now we are assuming F is C k smooth. Then, the map g again refer to the statement of the inverse function theorem to understand what this map g is then g is also C k smooth. In the proof of the inverse function theorem this g we have shown is actually going to be a C 1 smooth map.

Now, the assertion is that it is a C k smooth map ok. How does this proof go? Recall that Dg y is nothing, but DF at the point g of y inverse; this was 1 of the conclusions of the inverse function theorem, the derivative of the inverse is nothing, but the inverse of the derivative ok.

Now, our hypothesis is that the partial derivatives, the partial derivatives, derivatives of the components of F exist up to order k, up to order k and are continuous. This is essentially our hypothesis on the map F saying that F C k smooth means all the coordinate mappings are going to be going to have a partial derivatives up to order k and all these partial derivatives are continuous. Now, how can we use that well we now move to matrices?

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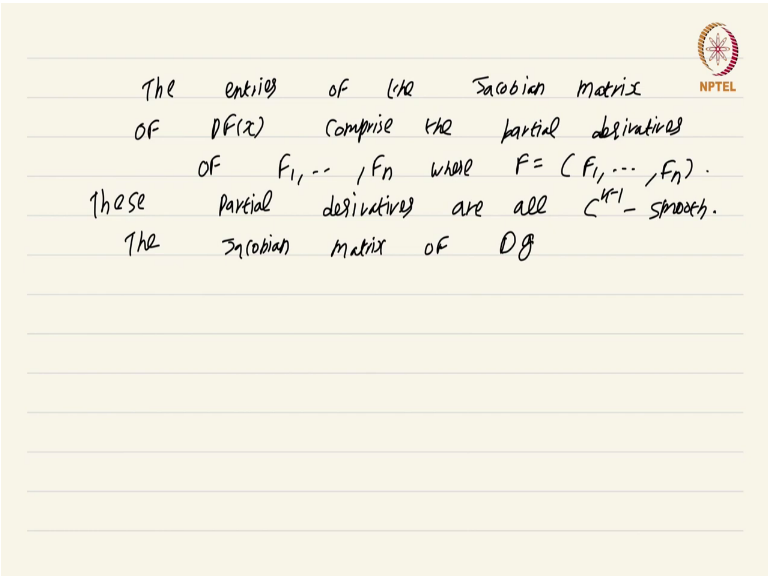

So, the entries, entries of the Jacobian matrix, the Jacobian matrix, we call the Jacobian matrix is nothing, but the matrix of partial derivatives it is nothing, but the matrix representation of the derivative map. So, the entries of the Jacobian matrix of D F x comprise the partial derivatives, the partial derivatives of F 1 to F n where, where F is nothing, but F 1 to F n.

These partial derivatives; these partial derivatives, derivatives are all C k minus 1 smooth by hypothesis. Because the functions F 1 to F n are C k smooth the partial derivatives have to be C k minus 1 smooth ok. Now the Jacobian matrix, of g of D g or rather here itself, it is not the Jacobian matrix of a D F x right, that is not the correct way to say it.

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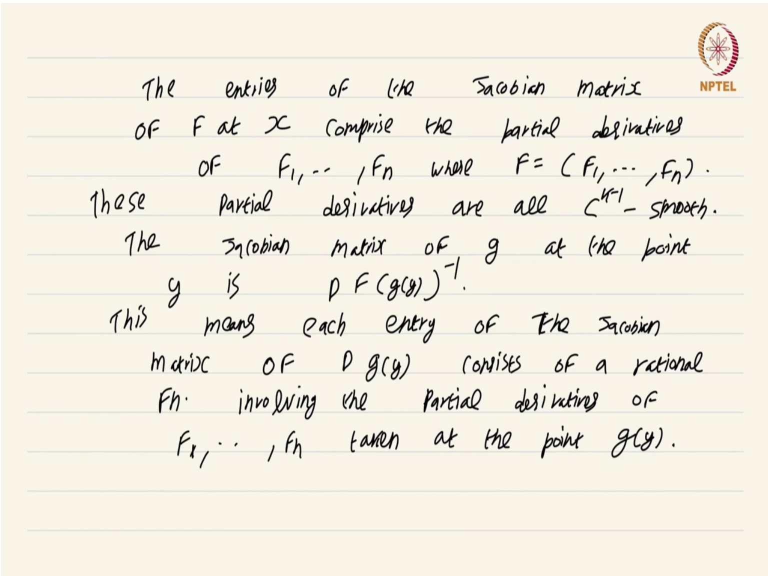

The entries of the Jacobian matrix of F at x. The Jacobian matrix is nothing, but the matrix of partial derivatives, which is nothing but the matrix representation of the derivative it. Does not make sense to say the Jacobian matrix of D F x that makes no sense.

So, in a similar way I must say the Jacobian matrix of g at the point y, at the point y is DF at the point g of y inverse this is the conclusion of the inverse function theorem, I am I am writing this again ok. This means each entry of the Jacobian matrix of Dg y consists of a rational function involving, the partial derivatives, the partial derivatives of F 1 to F n, F 1 to F n taken at the point g of y right?

So, how do we evaluate this D F g of y inverse, when you already know what D F is and you know, how to invert a matrix. It is just you will have to divide by the determinant of that matrix and take the co factor matrix or whatever I really do not remember. You have to take the co factor matrix of the various I mean blocking out the I th column and the j th row and taking whatever, you will have a better idea what the co factor matrix matrices are?

So, when you actually think about this what you are essentially doing is each entry will have some polynomial function of the various entries divided by the determinant, which itself is a polynomial in the entry. So, this is going to be a matrix each of whose entries, each of whose entries of this matrix involves a rational function with the denominator not 0, because that is part of the hypothesis. This matrix DF at the point g of i is invertible ok. So, the entries of the Jacobian matrix, again I made the same technical error of g at y of g at y.

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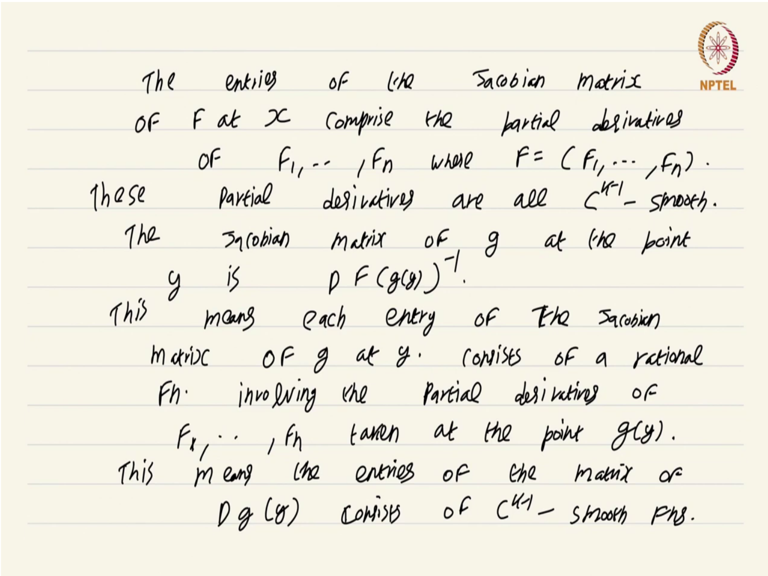

So, the Jacobian matrix of g at y consists of rational functions involving the partial derivatives and the denominator is the determinant, which is far away from 0 ok. Now, what does this say this means, the entries of the matrix of Dg y consists of C k minus 1 smooth functions. Why is it the case that the entries of Dg y consists of C k minus 1 smooth functions, well note that the partial derivatives of F 1 to F n are all C k minus 1 smooth, that we have talked about repeatedly. And, these partial derivatives are going to be evaluated at the point g of y.

So, far we have got g is just a C 1 smooth function. So, it does not seen obvious that Dg y the entries of Dg y is going to consist of just C k minus 1 smooth functions. Plus remember that there is a denominator involving the determinant and that function is also going to have a g is going to play a role, there also because the determinant is going to be evaluated at the point Dg g of y.

Therefore, it is not really clear that the entries of Dg y consists of C k minus 1 smooth functions, but we can prove this by induction ok.

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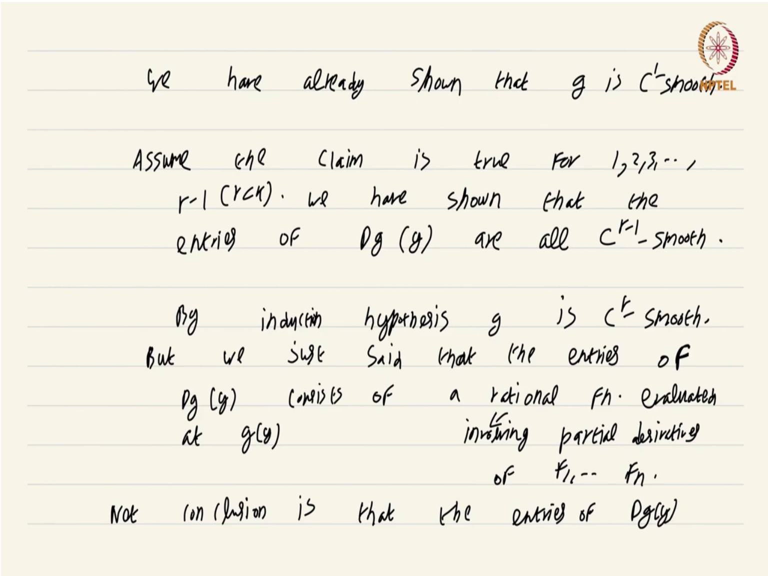

We can prove this by induction, well if I mean the base case is already done, we have already shown, we have already shown, already shown that g is C 1 smooth, that bit is already done. Therefore, the entries of the matrix Dg a is going to consist of continuous functions. So, the base k is done.

Assume that g is no the assumption is not on g assume that the result the claim is true for 1, 2, 3 dot dot dot r minus 1 less than k ok. That means, we have shown, we have shown, we have shown that the entries of Dg y are all are all C r minus 1 smooth ok. So, 1 correction I have to assume that r is less than k naught r minus 1 less than k because our hypothesis is that g is a C k smooth mapping.

So, the entries of D g we have to show is C k minus 1 ok. So, we have now, we have now shown; that means, not shown we have now by induction hypothesis, by induction hypothesis g is C r minus 1 smooth. In fact, it is C r smooth sorry about that, it is C r smooth, because we are assuming the by induction hypothesis that the entries of Dg y are all C r minus 1 smooth ok.

But we just said, we just said, that the entries, that the entries of Dg y consists of a rational function, evaluated at g of y. And, this is not just any whole rational function rational function is involving partial derivatives of F 1 to F n. But, the partial derivatives of F 1 F n are all C k minus 1 smooth and we are evaluating at a C r smooth function. The composition of a C k minus 1 smooth function and a C r function is continuous to be C r smooth, that is where we have assumed that r is less than k ah.

So, net conclusion is that the entries of Dg y entries of Dg y are all C r smooth C r smooth therefore, g is C k smooth by induction ok.

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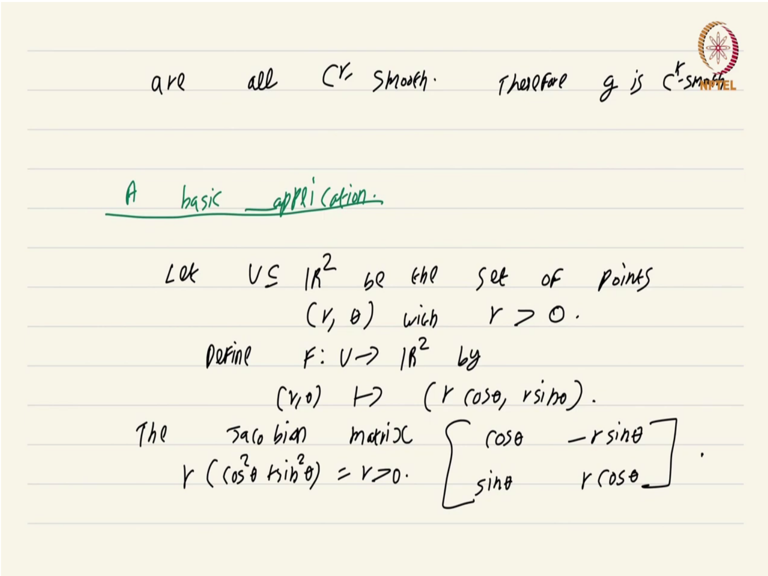

So, there is really nothing happening in this proof just writing it out just complicates the whole thing. Just think about what is happening? Think about how the Jacobian matrix of F is going to look think about how the Jacobian matrix of g is going to look, think about how what happens when you take inverse and this result should be fairly obvious ok.

So, we are now going to give a basic application of the inverse function theorem to polar coordinates and that will conclude this video so, a basic application. The rest of the course is really an application of the inverse function theorem. So, I do not want to focus too much on artificial numerical examples involving the inverse function theorem.

You should work out 1 or 2 they are there in the exercises, but I mean other than getting a feel for what is happening, they are not going to really give you insight into the in inverse function theorem. But, it is important to work out at least 1 or 2 just to make sure that you understand the mechanics of the inverse function theorem.

So, the application is polar coordinates. So, let U subset of R 2 be the set of points set of points r theta r theta with r greater than 0 ok. Define, F from U to R 2 by r theta maps to r cos theta r sin theta ok. So, this is nothing but transforming polar coordinates into usual Cartesian coordinates.

The Jacobian matrix, the Jacobian matrix a simple calculation will tell you that this is nothing, but cos theta, minus r sin theta, sin theta, then r cos theta ok. And, this will just be r into cos square theta plus sin square theta, which is equal to r, which is greater than 0.

So, the Jacobian matrix is going to be greater than 0 at all points r comma theta; that means, that this map F is a local C infinity diffeomorphism. You can extend the previous result C k smooth to k equal to infinity in a straight forward way. So, this essentially shows, that this transformation from polar coordinates to Cartesian coordinates is a local C infinity diffeomorphism.

You can actually write this down somewhat explicitly using r tan functions, you too would have definitely done in a basic course on multi variable calculus. But, without knowing I mean I am not familiar with how to write this down correctly, I mean, I have done it several times in my life, but you a if you ask me to do it instantly, I will probably make a mistake. But, nevertheless the inverse function theorem tells you that abstractly this map is going to be locally invertible.

So, that is the basic application of the inverse function theorem, you are going to see the more deep applications in the next video, where I talk about the implicit function theorem and then use the implicit function theorem to talk about many folds. This is a course on real analysis and you have just watched the video on diffeomorphisms and local diffeomorphisms.