**Real Analysis II**
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**Lecture - 17.1**
# The Inverse Function Theorem

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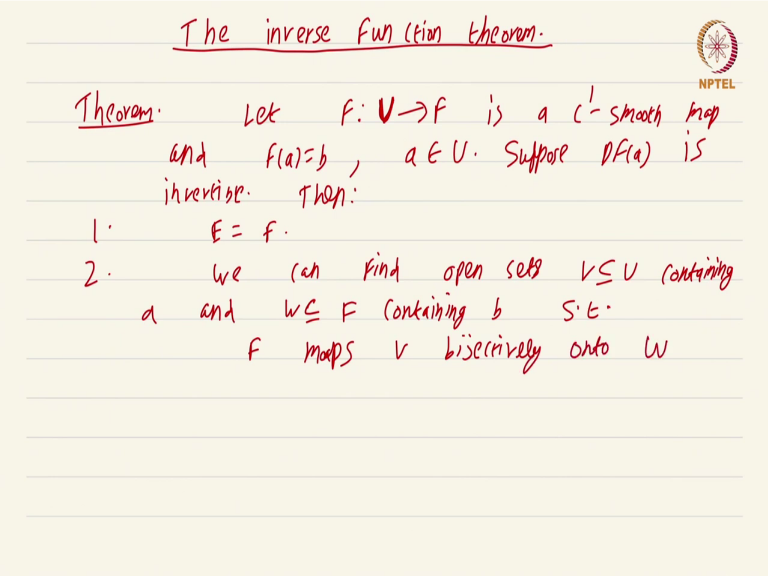


We now come to the inverse function theorem. The proof of this theorem involves a lot of nontrivial ideas and is probably the longest and toughest of this course. Not only is the theorem and proof hard its really useful. All this work will not go for go to waste. This theorem is extensively used in real analysis as well as other subjects like differential geometry. So also used in economics, physics, pretty much anywhere multivariable calculus is used, the inverse function theorem is definitely needed.

In this course, we will use the inverse function theorem to prove the implicit function theorem, which is really the starting point of the study of manifolds. For the rest of this course this theorem will keep popping up again and again. As the proof of this theorem is really involved let me make some remarks about the idea of the proof.

The basic idea of the proof is to somehow by hook or crook set up a situation where we can apply Banach's fixed point theorem or the contraction mapping principle. To do this we will use ideas inspired by Newton's method, but we will not quite use Newton's method or any convergence theorem in Newton's method to prove this theorem.

There are proofs available which use Newton's method to prove this for instance in Hubbard's textbook on multivariable calculus this is in fact, proved using Newton's method. So, those are some basic remarks let us state the theorem, even the statement of the theorem is a bit involved.

So, the setup is F from U to F; U to F is a C 1 smooth; C 1 smooth map and F of a equal to b, where a is a point in U. Now, the hypothesis, the central hypothesis is suppose DF a which is a linear mapping from E to F, suppose DF a is invertible is invertible. Then, we get a number of consequences, number 1 is the fact that E is equal to F this is obvious actually, the fact that a linear mapping is bijective between 2 vector spaces means, that both of them have to be isomorphic, but here E and F are both Euclidean spaces, so, they must be the same r n.

We can find open sets V subset of U containing a and W subset of F containing b, such that F of a is equal to b, sorry such that F of a equal to b is already given to us and W subset of F containing b such that F maps V bijectively onto W.

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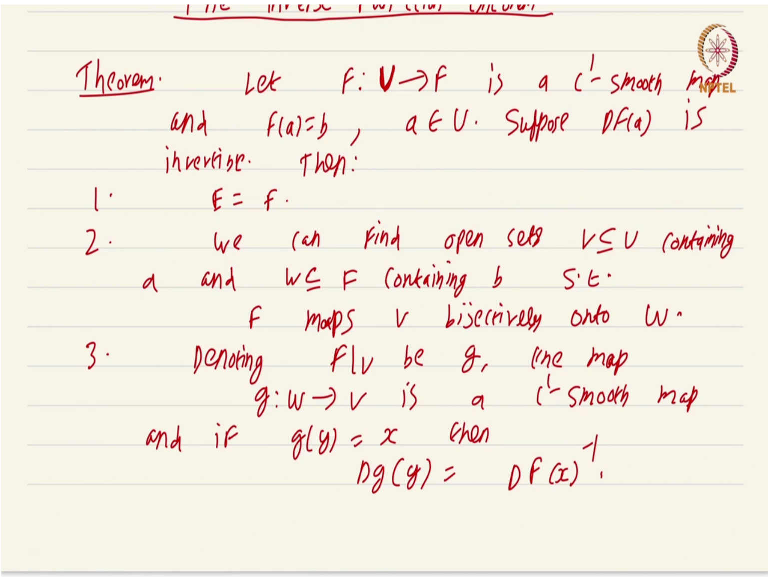


The third part is the most important conclusion, denoting F restricted to V by g, the map g g from W to V is a c 1 map is a c 1 smooth map and if g of y is equal to x, then Dg of y is nothing but DF x inverse ok.

So, this is the statement of the inverse function theorem. Now, let us first try to understand intuitively what the theorem is trying to say. Again the moral of the story is that if a function is differentiable, then the derivative is a good linear approximation of the function. This is something that you would have heard me say at least 568 times in this course. Now, note that we are given that the derivative map DF a is invertible.

So, as I said E equal to F comes as an easy consequence because DF a is an isomorphism therefore, F should also be r n. So, since DF a is invertible the natural conclusion to expect is that F is also invertible, because DF a is a good approximation of F, if DF a is invertible then naturally we would think that F is invertible, that is the vague loose idea.

But, remember that the derivative is an absolutely local thing. The derivative of F at a does not depend on the value of F slightly far away from a, it could be the values of F somewhat far away from a could be entirely different.

So, expecting that F will be globally invertible just because the derivative is invertible at a single point is a very very fools dream, that is not something that you can expect. In fact, I want you to think of a counter example to the stronger claim, that even if DF a is invertible for all points a in U, it is still foolish to expect F to be invertible ok. But, nevertheless the conclusion of the inverse function theorem is at least locally, we can find this open set V around a on which F is just a bijection.

Now, that is not the end of the story of the inverse function theorem. This inverse map g, this inverse map g is not just the set theoretic inverse; it is going to be a c 1 smooth map as well ok. We will set up some terminology regarding this in the next video, but for the time being you can think of F being invertible in a differentiable way. And, we can compute the derivative at the points y in w ok. So, this is a the basic intuition behind the inverse function theorem.

If you have not already done so, I urge you to please watch that short video on the inverse function theorem in one variable, which was actually a part of the previous course on real analysis. Now, let us dive into the proof as I said there is a lot of work involved in this proof. So, let us go full speed ahead ok.

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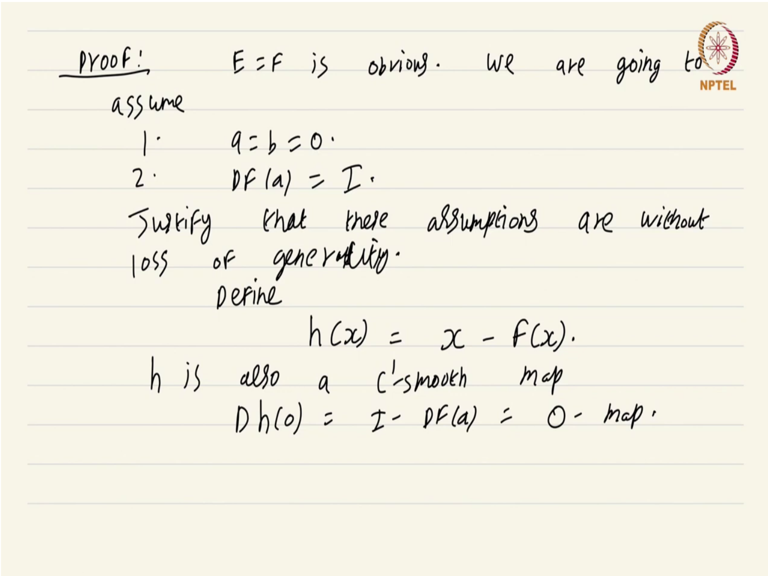


Proof, so, I am going to make some simplifications first is E equal to F is obvious. Now, the simplifications I am going to make are the following, we are going to assume; going to assume first of all that a equal to b equal to 0 and second D F a is the identity matrix ok.

Now, I want you to justify these assumptions, justify that these assumptions, these assumptions are without loss of generality ok. What I mean by that is if you can prove the inverse function theorem, in this setting where a equal to b equal to 0 and the derivative map is just the identity. The general result when a is not equal to b and may not be equal to 0 and DF a is just an invertible a map and not the identity map, this really does not matter ok; the general case will follow very easily ok.

So, the first step is to define a new function. Define h of x, h of x equal to x minus F of x ok. Then, observe that h is also a c 1 map, h is also a c 1 smooth map, but the interesting thing is by our assumptions Dh 0 is nothing but identity minus DF a which is just the 0 map ok. So, this is the interesting bit the derivative of h at 0 is nothing but the 0 map.

(Refer Slide Time: 09:20)

By continuity of Dh, we can find a small ball $B(0,r) \subseteq U$ s.t.

$$\| Dh(x) \|_{op} \leq \frac{1}{2} \quad \forall x \in B(0,r).$$

On $B(0,r)$, we have

$$\| I - DF(x) \|_{op} \leq \frac{1}{2}$$

In an earlier example where we studied the derivative of inverses (matrices). We know that $DF(x)$ is invertible on $B(0,r)$.

Now, we observe that by continuity of Dh, we can find; we can find a small ball small ball B 0, r subset of U such that, the operator norm of Dh of x is less than or equal to half for all x coming from this ball. This follows from the continuity of Dh, to understand why this is the case you will have to understand this coordinate independent formulation of the derivative being continuous right.

We had defined h being c 1 just to mean that the partial derivatives exist and are continuous. But in a video earlier I had also talked about how, we can talk about continuity of the derivative map in terms of treating it as a map into the space lef with the operator norm.

So, that is the reason, why we can find a small ball such that Dh x Dh at x the operator norm will be less than or equal to half for all points ok. Now, on B 0 r, we have the operator norm of identity minus the derivative of F at x is less than or equal to half, this is just rewriting. What are the derivative of h at a given point x is it is just a derivative of x minus F of x its identity minus DF of x ok.

Now, in an earlier example, where we studied, where we studied, the derivative of inverses, derivative of inverses this is in the context of matrices. This is in the context of matrices we studied this matrices and linear mappings are just the same thing represented in different ways. We had studied the derivative of the inverse map on matrices, in an earlier example this was the long involved example.

We have already seen that if you have this identity minus DF x op is a less than or equal to a 1. In fact, less than 1, we know that DF x is going to be invertible, we know that DF x is invertible on B 0, r ok. Because, the operator norm of identity minus DF x is not very large the invertibility of F continues for all points x in B 0, r ok.

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Fix $x \in B(0,r)$

$\|h(x)\| \le \frac{1}{2}\|x\|$ the mean value theorem

applied on the segment joining $0$ to $x$.

$h$ maps $\overline{B}(0,r)$ into $\overline{B}(0,\frac{r}{2})$.

Claim: If $y \in \overline{B}(0,\frac{r}{2})$ then $\exists$ unique $x \in \overline{B}(0,r)$ s.t. $F(x) = y$.

Now, fix x in B 0 r, we have an estimate about the operator norm of the derivative of h we know that it is less than or equal to half. So, you can apply the mean value theorem and conclude that norm h of x, norm h of x is less than or equal to half norm x, this is just the mean value theorem, this is just the mean value theorem, applied on the segment, on the segment joining 0 to x. Remember h of 0 is also 0 ok.

So, this is just the mean value theorem applied to the segment joining 0 to x, we get that norm h x is less than or equal to half norm x. This just means that h maps, the closed ball of radius r, the closed ball of radius r into the closed ball of radius r by 2 ok. We now claim, we now make a claim, if y is an element of B closure r by 2, then there x is unique, there x is unique x in B closure 0 r, such that F of x is equal to by note not h of x equal to y, but F of x equal to y.

So, I am not claiming that F maps B closure 0 r into B closure r by 0 r by 2 h already does that, h takes B closure 0 r into B closure 0 r by 2, but what I am claiming is that if you look at the set of points, that are mapped into B closure 0 r by 2. But, that which come from B closure 0 r every single point in B closure 0 r by 2 will be mapped ok by F.

So, the claim is given y in B closure 0 r by 2 there is a unique point x in B closure 0 r such that F of x equal to y ok. So, here is the part where we will be we will take our inspiration from Newton's method.

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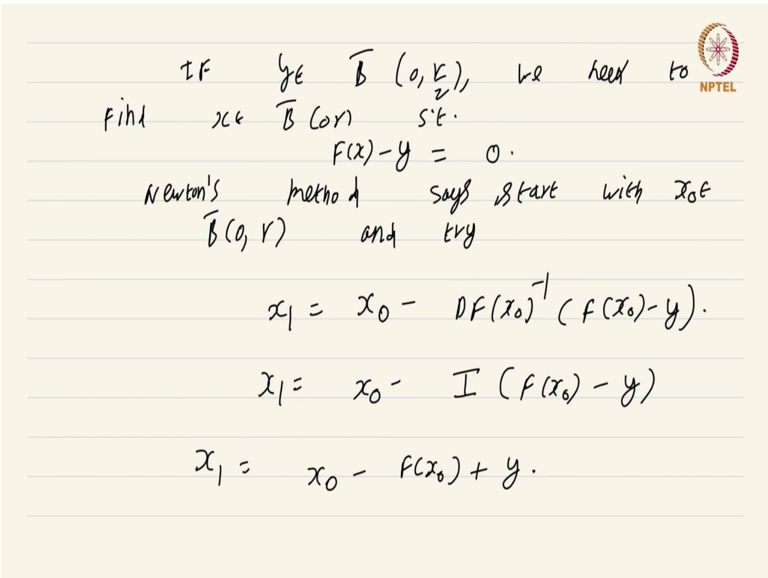


So, what we want to do is if y is in B closure 0 r by 2 we need to find, we need to find x in B closure 0 r such that, F of x is equal to y, this is what we want to do or we will rewrite this slightly. Such that F of x minus y is equal to 0, this is what we want to do ok.

Now, Newton's method says, start with some x naught in B closure 0 r and try x 1 is equal to x naught minus DF x naught inverse acting on F of x naught minus y. So, here we are fixing y. So, we are fixing y and we want to find a solution of F of x minus y equal to 0 we want to solve this equation, Newton's method says just start with the guess and look at the better guess x 1 equal to x naught minus D F x naught inverse F of x naught minus y.

Now, here we are going to pull a trick we already know that DF 0 inverse is going to be equal to the identity. And, we also know from what we have considered previously, we also know that this mapping not this mapping this derivative identity minus DF x the operator norm is less than or equal to half. Its not too far away from being the identity, whenever you choose x from this B 0, r closure ok.

So, what we are going to do is we are going to try our luck and we are going to simplify this expression, by saying we want to guess x 1 equal to x naught minus just the identity. F of x naught minus y. You might wonder how somebody thought of replacing DF x naught inverse by the identity mapping; well you will understand in a moment that this actually makes our job really easy. But apart from that whenever you have a sophisticated theorem, the final proof that you see nowadays is a modern proof that has been obtained over the years.

So, it is not as if somebody got this without any trial and error or experimentation. A fair bit of experimentation would have happened and finally, just treating D F x naught inverse as just the identity map seems to work. So, just hold on for a few minutes you will see that this is actually a good idea.

So, once you substitute identity you get the simplification that x 1 is x naught minus F x naught plus y. Again, I remind you that y was fixed we are trying to find a point x and B 0 are closure such that F of x is equal to y. And, this is our better guess of the solution, we start with an initial guess x naught and we say the better guess is x naught equal to x naught minus F x naught plus y.

So, this is just our intuition that comes from Newton's method. Now, what we are going to do is we are going to turn this inspiration into a rigorous proof in the following way.

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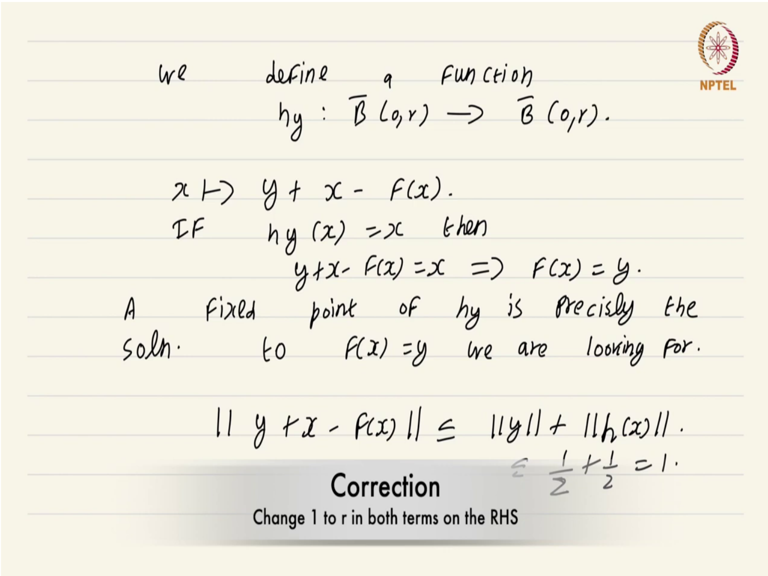


So, we define a function, we define a function, which we call h y this function starts from B 0, r closure and I will just leave the co domain blank for the time moment, this is defined by x maps to y plus x minus F of x. And, you can clearly see where the inspiration for this particular choice of function comes from ok.

Now, this function has this nice property that if h y of x equal to x, then that just means that y plus x minus F of x equal to x, which just means that F of x equal to y. In other words a fixed point of h y is precisely the solution we are looking for, the solution to F of x equal to y we are looking for. So, we have translated finding the solution to finding a fixed point this seems promising, because just a while back we studied something called the Banach fixed point theorem.

So, unless this is all happening in a miraculous way things should work out ok. Now, let us see what the co domain of this function h y is going to be. Well to see that observe that norm y plus x minus F of x is just less than or equal to norm y plus norm h of y, this is not h subscript y sorry about that this is norm of h of y. This is just because the function x minus F of x, if you sorry this is not h of y, this is h of x h of x.

So, this just follows because whenever for a given point x the function h of x is just x minus F of x and we have just used the triangle inequality. Now, note that this point y is also coming from the ball of radius half closure, and h of x is also going to be a point in the ball of radius half closure. Simply because we already saw by that mean value theorem that h maps B 0 are closure inside B 0, r by 2 closure. So, this is less than or equal to half, plus half which is equal to 1 ok.

In other words in other words this blank that I had left thankfully turns out to be B 0, r closure ok. And, the required solution of F of x equal to y is nothing but a fixed point of this map h y we have a self map and B closure 0 r is a complete metric space being a compact subset of r n. Therefore, all the stage, all the parts are in place we just have to fix it together and apply the Banach’s fixed point theorem and all we have to do is to somehow show, that this map h y is a contraction.

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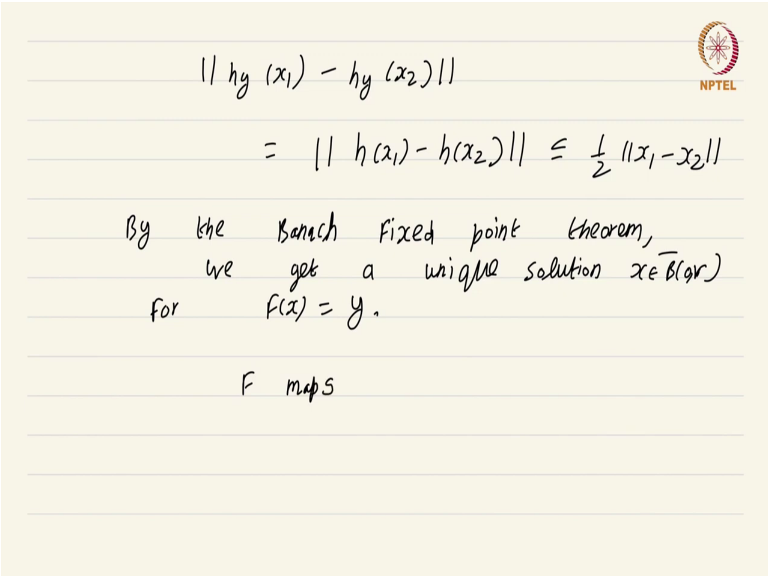


And, that is not at all hard what we have to do is we have to take norm of h y of x 1 minus h y of x 2 ok, where x 1 and x 2 come from B 0 are closure. And, you can just check by the formulas that this is nothing but h of x 1 minus h of x 2 ok. And, this is less than or equal to half of norm x 1 minus x 2 ok.

So, this part just follows, this part just follows by the application of the mean value theorem yet again ok. Now, by the Banach fixed point theorem, Banach fixed point theorem point theorem, we get a unique solution, we get a unique solution x in B 0, r closure. Of course, I have written the closure in an extremely bad way this is B 0, r closure, for F of x equal to y excellent ok.

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$$\|h_y(x_1) - h_y(x_2)\|$$

$$= \|h(x_1) - h(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|$$

By the Banach Fixed point theorem, we get a unique solution $x \in \overline{B}(0, r)$ for $F(x) = y$.

Every point in $\overline{B}(0, \frac{r}{2})$ has a Pre-image under $F$ in $\overline{B}(0, r)$.

So, what is the net upshot, we know that F of F maps or rather let us write it correctly, every point in B 0, r by 2 closure has a pre image under F in B 0, r closure. Note, we have not shown that this B 0, r closure maps into B closure 0 r by 2. It could map certain points of B 0, r closure outside of B 0, r by 2 closure. But at least we get a unique solution for any given point y in B 0, r by 2 closure there is exactly 1 x in B 0, r closure such that F of x equal to y ok.

So, what we do is the following? We try to find out the set V that we require such that F restricted to V is going to be a 1 1 and onto an open set W in F that is one of the parts of the inverse function theorem that we have to show. So, let us just since it is been quite some time let us just look at the statement, we have to find this open set V subset of U containing a and W subset of F containing B such that F maps V bijectively onto W ok.

So, this is our objective we are going to use this fact that F as solutions in B 0, r by 2 closure to manufacture our V.

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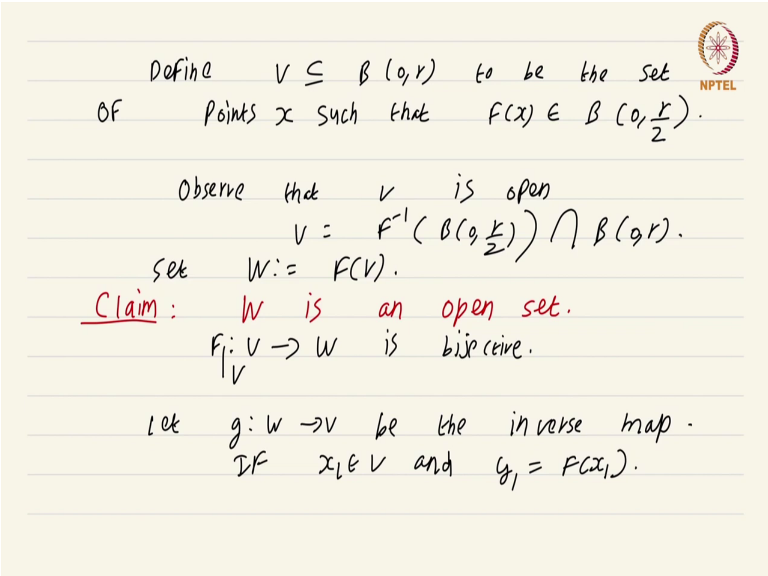


So, define V subset of B 0, r to be the set of points, set of points such that. So, set of points let us say x such that F of x is an element of B 0, r by 2 ok. So, we already know that, when we consider the collection of all points in B 0, r closure and look at F of that, then you would be you would get a set that is a super set of B 0, r by 2 closure. Now, we look at those points in B 0, r which map inside B 0, r by 2 ok.

Now, observe that V is open V is open this bit is actually obvious because V is nothing but F inverse of B 0, r by 2 which is an open set because F is continuous intersect B 0, r to make sure that you do not pick up any points outside B 0 r. So, clearly V is open. Set W to be F of V right.

So, first of all claim, another claim, the second claim of this proof W is an open set, W is an open set ok. Now, this claim is somewhat tricky to prove. Let us a get to it right away. We know that, F from V to W is bijective ok, to be 100 percent precise I must write F restricted to V from V to W is bijective this is just the way we manufactured V and W, it is certainly onto because W is F of V, it is certainly injective simply because of the uniqueness part of the previous claim ok.

Now, to show that this is an open set, what we are going to do is we are going to consider the inverse already let g from W to V be the inverse be the inverse map ok. Now, our aim is to somehow show that if there is a point such that so, if x 1 is a point in v and y 1 equal to F of x 1 ok. Our aim is to show that there is an open set around this point y 1, which is fully contained in W or more precisely if we can find some ball around y 1, which is fully contained in W, then we are actually done ok. So, we have fixed these points x 1 in v and y 1 is in F of x 1.

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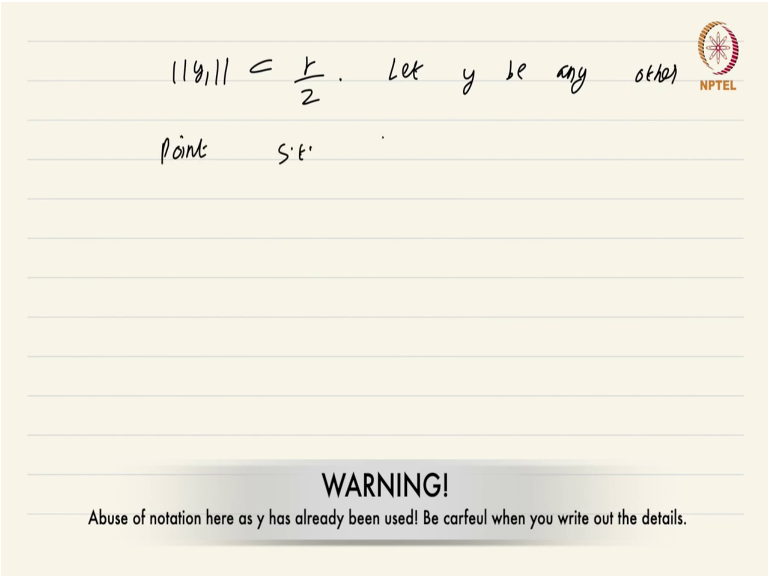


Now, by definition norm of y 1 is less than r by 2 ok. Now, let y be any other point, any other point, any other point, such that norm y is also less than r by 2 ok.

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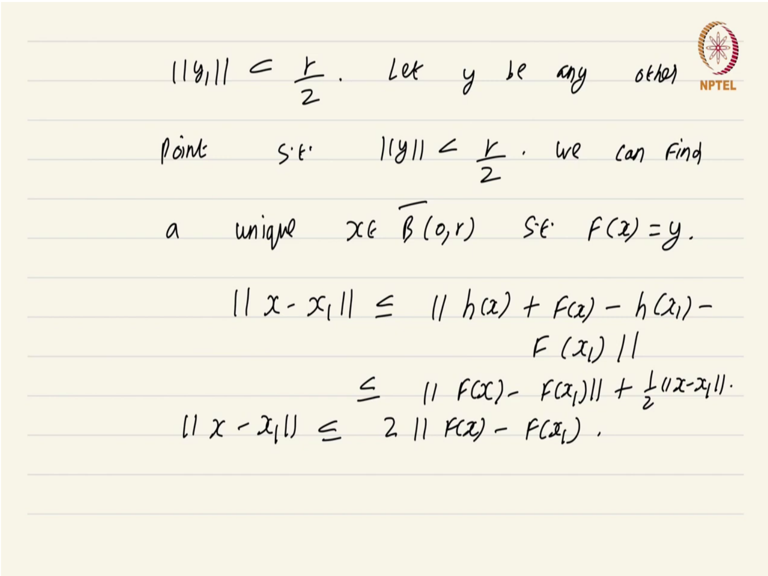


Then, we can find; we can find by the previous claim we can find a unique x in B 0, r closure such that F of x equal to y right. Now, observe that norm of x minus x 1 is less than or equal to norm of h of x plus F of x minus h of x 1 minus F of x 1. This is just because h of x is defined to be x minus F of x and there is plenty of cancellations that happen and this is this will just drop out ok. In fact, I believe this is going to be an equality not even a less than or equal to.

So, this is less than or equal to norm F x minus F x 1, this is by triangle inequality plus half norm x minus x 1. Because h is a contraction, h is a contraction we have already seen this is the just the application of the mean value theorem. This is just going to be norm F x minus F x 1 plus half norm x minus x 1 ok.

So, the net upshot of all this is that norm x minus x 1 is less than or equal to twice norm F of x minus F of x 1 ok. So, the we just take half of norm x minus x 1 to the other side and cross multiply this 2 and you get this ok.

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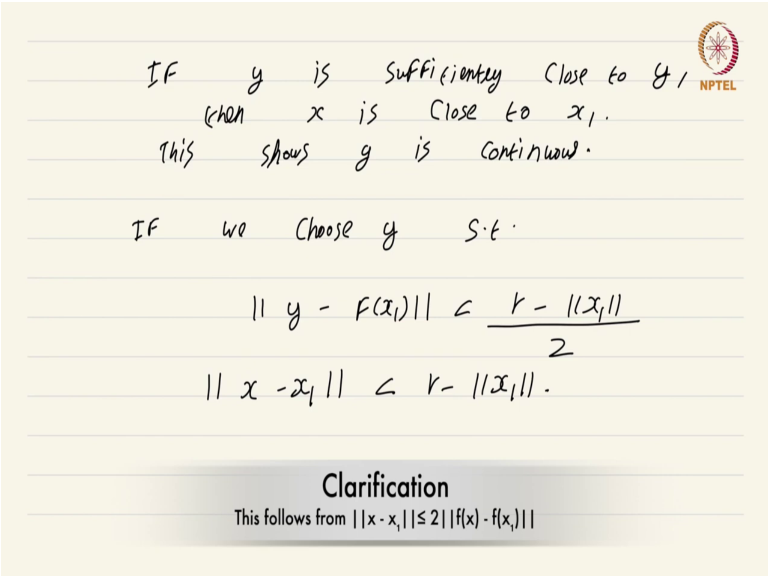


So, what does this show? This shows that, if y is sufficiently close to y 1, then x is close to x 1, because norm x minus x 1 is less than or equal to twice norm F of x minus F of x 1. This shows g is continuous right. The previous inequality actually shows that g is continuous, g remember was the inverse of F restricted to V ok, excellent.

Now, if we choose, if we choose y such that norm y minus F x 1 is less than r minus norm x 1 by 2. Suppose, we choose y in such a way that y minus F x 1 norm is less than r minus norm x

1 by 2. Then, what happens is norm x minus x 1, norm x minus x 1, will be less than; will be less than r minus norm x 1 ok.

So, if you choose y such that norm y minus F x 1, is less than r minus norm x 1 by 2 then norm x minus x 1 will be less than r minus norm x 1.

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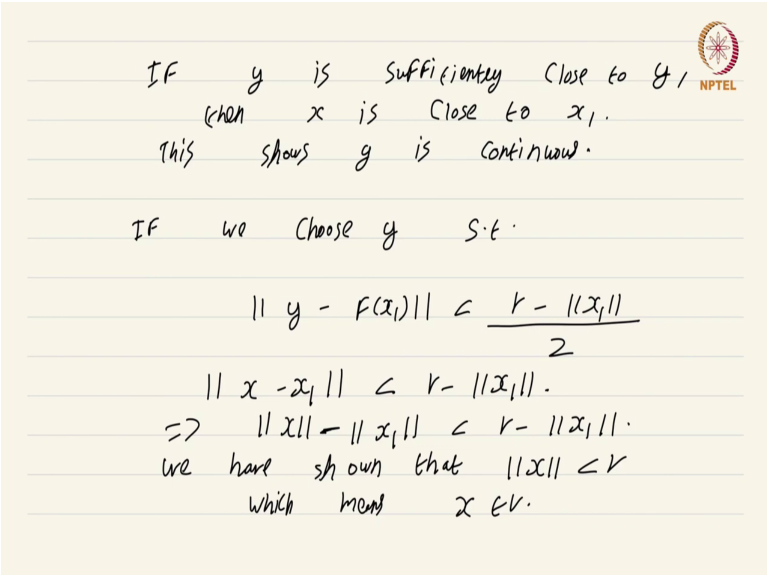


Which just means that this means that norm x minus norm x 1 is less than r minus norm x 1. Where we have just applied this reverse triangle inequality ok. So, what this shows is that we have now shown, we have shown, we have shown, that norm x is therefore, less than r just cancelling of the norm x ones.

We have shown that norm x is less than r. So, what this shows finally, what this shows finally, is the fact that this shows that W is open ok. This shows that W is open.

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This shows that $W$ is open because $F(x) = y \in W$.

We have to show $g$ is differentiable and $C^1$- smooth on $W$. We have chosen $r$ s.t.

$$\|Dh(x)\|_{op} = \|I - DF(x)\|_{op} \leq \frac{1}{2}$$

whenever $x \in B(0,r)$.

$DF(x)$ is invertible whenever $x \in B(0,r)$. Let $y, y_1 \in W$ and choose $x, x_1 \in V$ s.t. $F(x) = y$, $F(x_1) = y_1$.

So, what are we actually obtain? What we have shown is if you choose the ball of radius r minus norm x 1 by 2 centered at the point F x 1, then if you choose any point y in this ball then the corresponding pre image x satisfies norm x less than r, which simply means that x is in v.

So, I mean I skipped a step I was a bit hasty we have shown that norm x less than r which means, x is in v x is in v ok. This shows that W is open, because F of x equal to y is an element of W ok. So, what is the net upshot of all this w is also open ok? So, we have achieved quite a bit, if you go back to the statement of the inverse function theorem which was a long time ago, we have reached till the fact that this map g from W to V is a continuous map and it is a bijection.

So, we still have to prove that it is a c 1 smooth map and also compute its derivative ok. So, we have to show that g is differentiable; we have to show that g is differentiable ok. So, let us show, we have to show, we have to show g g is differentiable and. In fact, c 1 and c 1 smooth on W ok.

Now, we have chosen long back we have chosen r such that the norm of h which you recall was the map x minus F of x. This operator norm is nothing, but I minus DF x operator norm is less than half, whenever I think less than or equal to half I do not remember, whenever x belongs to B 0, r.

This is the way we have chosen this r. Again we had already concluded from this that D F x is invertible is invertible, whenever x is in B 0, r, whenever x is in B 0, r ok. Now, what is the objective now we have to show that g is differentiable. So, let y comma y 1 be points of W and choose x comma x 1 in v such that F of x equal to y and F of x 1 equal to y 1 ok.

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$$g(y) - g(y_1) - DF(x)^{-1}(y - y_1)$$

$$= \frac{x - x_1 - DF(x)^{-1}\left(F(x) - F(x_1)\right).}{\|y - y_1\|}$$

We have to show sublinearity of above term.

Now, our goal is to analyze g of y minus g of y 1 ok. And, our candidate derivative claimed is nothing, but DF x. So, what we do is we subtract DF x inverse of y minus y 1 ok. So, since the candidate derivative is DF x inverse, we subtract this and somehow we want to show that this is a sub linear function. Now, by our choice of notation this is just nothing but x minus x 1 minus DF x inverse of x F of x minus F of x 1 ok.

Now, we have to show that this entire term. This entire term is sub linear ok. We have to show, we have to show sub linearity of the above term linearity of above term.

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$$g(y) - g(y_1) - DF(x)^{-1}(y - y_1)$$

$$= \lim_{y_1 \to y} x - x_1 - DF(x)^{-1}(F(x) - F(x_1)) = 0$$

We have to show sublinearity of above term.

In other words what we actually have to show is we have to show, that this whole thing divided by norm y minus y 1 and limit y goes to y 1 goes to y is 0 ok. That is what we essentially have to show ok.

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$$g(y) - g(y_1) - DF(x_1)^{-1}(y - y_1)$$

$$= x - x_1 - DF(x_1)^{-1}(F(x) - F(x_1)).$$

We have to show sublinearity of above term.

By the differentiability of F,

$$F(x) - F(x_1) = DF(x_1)(x - x_1) + E(x - x_1).$$

This gives:

$$x - x_1 - DF(x_1)^{-1}(y - y_1) = - DF(x_1)^{-1} E(x - x_1).$$

Now, by the differentiability of F by the differentiability, by the differentiability of F, we can write, we can write F of x minus F of x 1 this is the term that occurs as part of what we have to show. F of x minus F of x 1 this is just DF x x minus x 1 plus E of x minus x 1 ok.

This is just coming from the differentiability of F and E of x minus x 1 is nothing, but the error term ok. So, what do we have to do? We have to now we have to now analyze this. So, this gives x minus x 1 minus DF x 1 inverse, DF x 1 inverse, if you do not mind instead of showing differentiability at the point y, I will show it at y 1 because the I said let us show differentiability of F at the point y, but y and y 1 were arbitrary. So, the way I have written it out it makes more sense to do it at the point y 1.

So, if you do not mind let me just make a minor change, this is just x 1 inverse, and this is just again x 1 inverse ok. And, again here also this is just x 1 inverse; this is just x 1 not x 1 inverse ok. So, this gives x minus x 1 minus DF x 1 inverse of acting on y minus y 1 is equal to is equal to minus DF x 1 inverse acting on, acting on the error term x minus x 1 ok. How did we get this? Well x minus x 1 minus DF x 1 inverse of F x minus F x 1 we just substituted this DF x 1 x minus x 1 plus E x minus x 1 here.

And, then DF x in and a DF x 1 inverse both of these just cancel off. And, you are just left with x minus x 1 plus plus DF x inverse DF x 1 inverse of E of x minus x 1 which is what I have written here ok right.

So, we have got this term and we have to estimate we have to estimate this right hand side and show that the right hand side is actually sublinear, except the sub linearity we have to show with respect to the y variable. Even though it is written as with respect to the x variable, we have to actually show it with respect to the y variable ok

So, what we have to get, what we have to show is this term, this term, let us take norm everywhere.

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$$\| x - x_1 - DF(x_1)^{-1}(F(x) - F(x_1)) \|$$

$$\leq \| DF(x_1)^{-1} \|_{op} \, \| E(x - x_1) \|$$

Take $y \to y_1$.

$$\frac{\| E(x-x_1) \|}{\| y - y_1 \|} \leq 2 \times \underbrace{\frac{\| E(x-x_1) \|}{\| x - x_1 \|}}_{\to 0}$$

as $y \to y_1$ $x \to x_1$ by continuity of $g$.

So, we have to show norm of x minus x 1 minus DF x 1 inverse of F x minus F x 1 we want to show sublinearity of this. This is by the previous line less than or equal to the operator norm of DF x 1 inverse times, the error term x minus x 1 ok. Now, take y going to y 1 take y going to y 1 ok. And, we have once y goes to y 1 we have norm of E x minus x 1 by norm y minus y 1, this is less than or equal to 2 times 2 times norm of E of x minus x 1 x minus x 1 by norm x minus x 1.

How did we get this norm y minus y 1 and relationship between this and this quantity? Well some time back we actually showed this, I do not seem to find where yeah here it is we have actually shown this some time ago. So, plugging that in we actually get that norm E of x minus x 1 by norm y minus y 1 is less than or equal to twice, norm E x minus x 1 by norm x minus x 1 and this goes to 0, because as y goes to y 1 x goes to x 1 by continuity of g by continuity of g ok.

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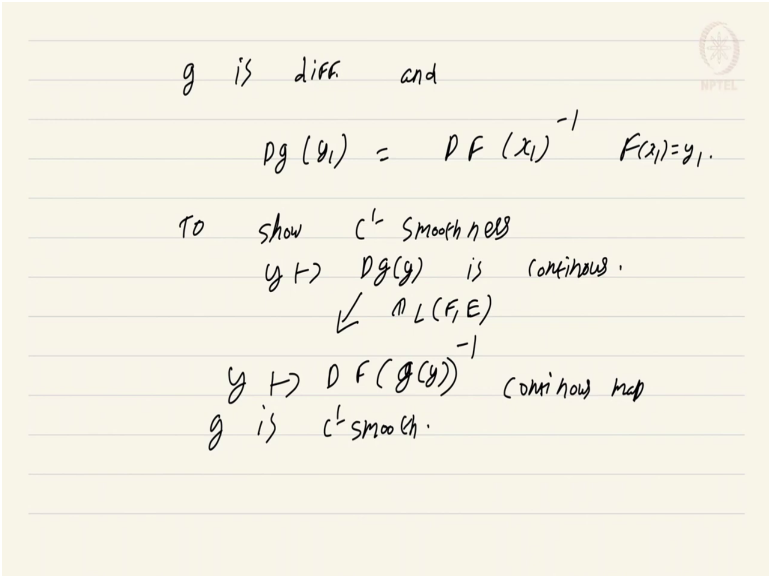


So, what is the net upshot, net upshot is g is differentiable and the derivative D g at y 1 is just D F at x 1 inverse, where F of x 1 equal to y 1. So, we are almost done almost everything is proved, we still have to show that this a map g is c 1 smooth, to show c 1 smoothness to show c 1 smoothness, we use that coordinate independent formulation of c 1 smooth. We just have to show that the mapping y goes to D g y is continuous. Where we treat where we treat the co domain as L E or L F E so, this is actually in l F e.

In this case it is just r in r n ok, but what we have is this map D g y is can be rewritten as D of F of g of y inverse ok. Now, g is continuous, g is continuous, F is certainly continuous, this whole thing is therefore, continuous and taking inverse is also continuous ok.

So, why is taking D here continuous? Because we have already assumed that F is a c 1 map and taking inverse is a continuous map, we have seen it long back we have in fact, shown that taking inverses, when treated as a map on matrices is in fact, a smooth mapping we have already seen that. When we did that example involving differentiating the inverse map.

So, DF g y inverse is in fact, c 1, this is in fact, continuous map, this is a continuous map y going to this. So, this shows g is c 1 smooth. So, all the part of inverse function theorem is now completely proved. So, please go through this lecture carefully again and please refer the lecture notes, where everything is done in a way that you can actually refer back and forth. Because of technological limitations I can display only a finite amount of mathematics on a given screen and that is going to make the proof difficult.

So, this I what I am saying in this lecture is just going to give you the basic idea of how this theorem is proved, it is up to you to now go through the proof line by line carefully in the notes. So, that you have a clear understanding of what is happening. This is a course on real analysis and this was the long video on the inverse function theorem.