Real Analysis II
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Lecture - 16.2
Newton's Method

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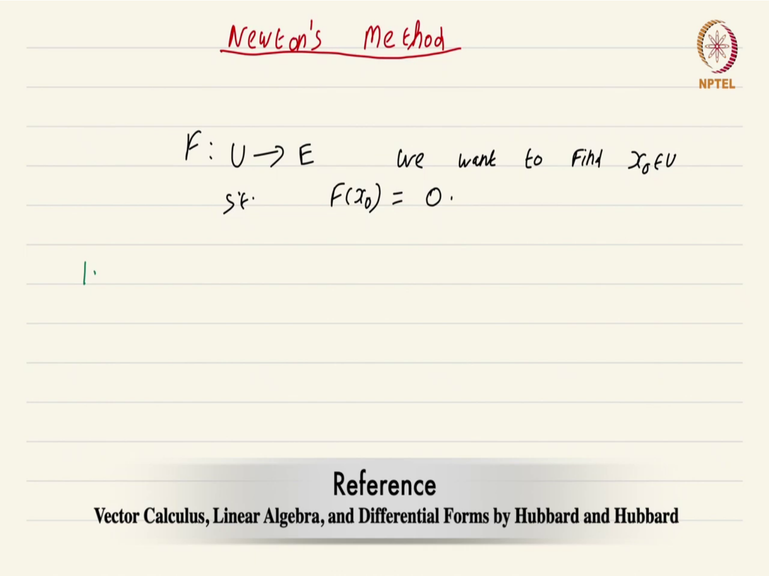

If you have been diligently watching this course till now, you would be now familiar with my usual spiel about how linearizing a problem is of great value. I repeatedly say that the central idea of this entire course is that non-linear phenomena are difficult and linear phenomena are much easier.

The derivative gives the best linear approximation. And now, we are going to use this to actually solve a concrete problem that often pops up in the real world. In fact, I am going to illustrate it with a commonly used algorithm for finding the square root, justify it using this linearizing idea.

So, this is really our first illustration of this idea, Newton's Method. Newton's method abstractly is as follows. You have a function F from U to E, and we want to find, we want to find x naught in U, such that F of x naught is equal to 0. In other words, we want to find a 0 of a given function F, ok. So, the abstract Newton's method can be described in a few simple steps which we are going to do now. We will just analyze a simple case in which Newton's method actually works and we will prove that it works. Hubbard's textbook gives more information about Newton's method.

Let me just say that Newton's method is till date not fully understood, we have only partial results. But for many many practical purposes it is very useful. So, here is the abstract Newton's method for solving this problem of finding a value, point x naught, such that F of x naught is 0.

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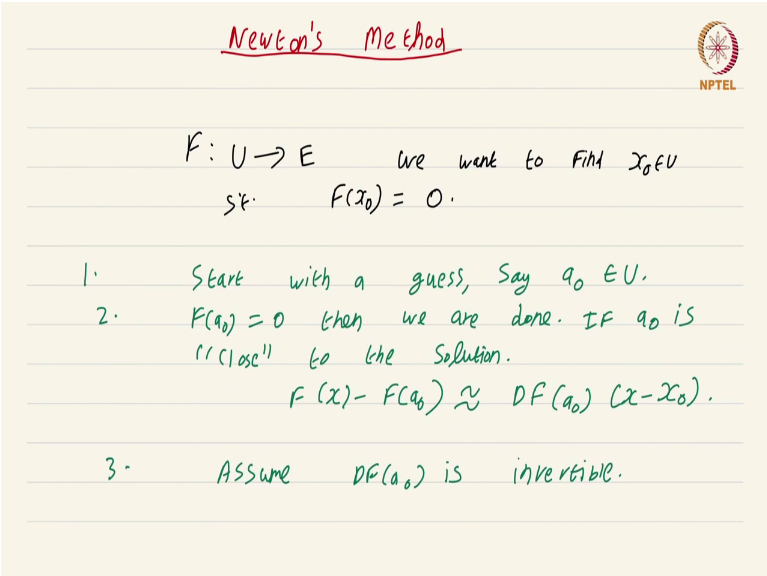

Start with a guess start with a guess say, a naught, ok. So, step 2 is if F of a naught is equal to 0, then we are done, ok. But this is not going to happen in the real world, no one is that lucky. Now, we know that if a naught is close to the solution, that is if you have chosen a naught in such a way that it is not too far away from the actual solution, we know that F of x minus F of a naught is approximately equal to DF at a naught times x minus x naught, where this x is supposed to be that hypothetical solution, ok.

Now, 3rd part assume DF at a naught is invertible, ok. Then, what you can do is, you can pull this trick of taking DF a naught to the other side, ok. We can pull this trick of taking DF a naught to the other side and set F of x to be 0. Of course, x is supposed to be the place where the solution is happening.

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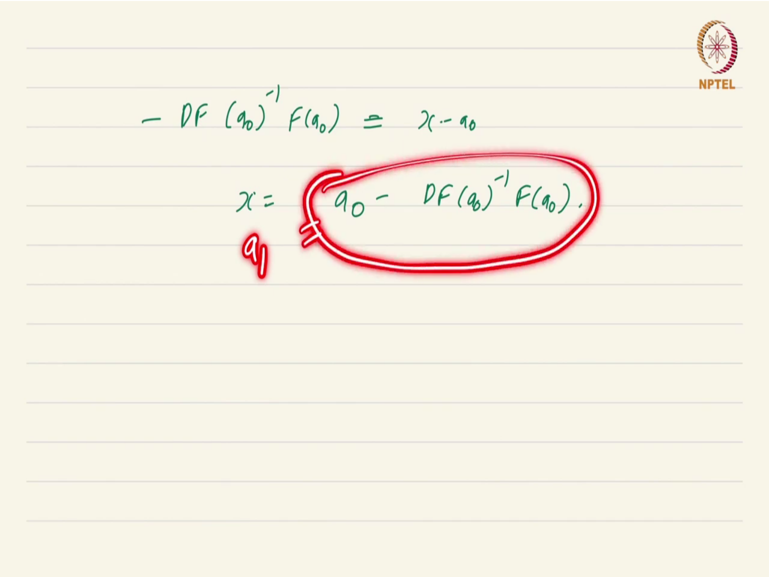

So, what we get is DF a naught, DF a naught inverse F of a naught negative sign is nothing but x minus a naught. Or in other words, x is equal to a naught minus DF a naught inverse F of a naught, ok. So, what Newton's method says a start with an initial guess a naught, start with this guess a naught, then what you do is you set a 1 to be this quantity, a naught minus DF a naught F a naught, call that a 1.

Then, plug in a 1 and see if F of a 1 is sufficiently close to 0. In most practical purposes, you are not really interested in exact solution. You want something that is very close to the actual solution that more than suffices. Check if F of a naught is very very very very close to 0, ok.

Of course, it might happen that F of a naught is really small, but a naught is nowhere near the actual 0, that can happen. A function can get very close to 0 and then decide to change its mind and go in a different direction that can happen. There is a real danger. All these intricacies have to be taken care of in a proper manner, if you are going to use Newton's method in a crucial application like launching a rocket or something like that.

In any case, you just iterate this procedure, till you are sufficiently satisfied that you are close to a solution. So, there is a theory behind Newton's method. You can prove a general theorem when Newton's method will work, but that is no, so far not the most ideal result. And this is the work of the Nobel Prize winning economist Kantrowitz. I will not go into that in this particular video. It is not needed. It is done in great detail in Hubbard's book, ok.

All I want to do is to take a very very special case and work out an example. The example is a classical one of taking square roots by this method called divide an average. Many calculators still continue to use this method for finding the square roots.

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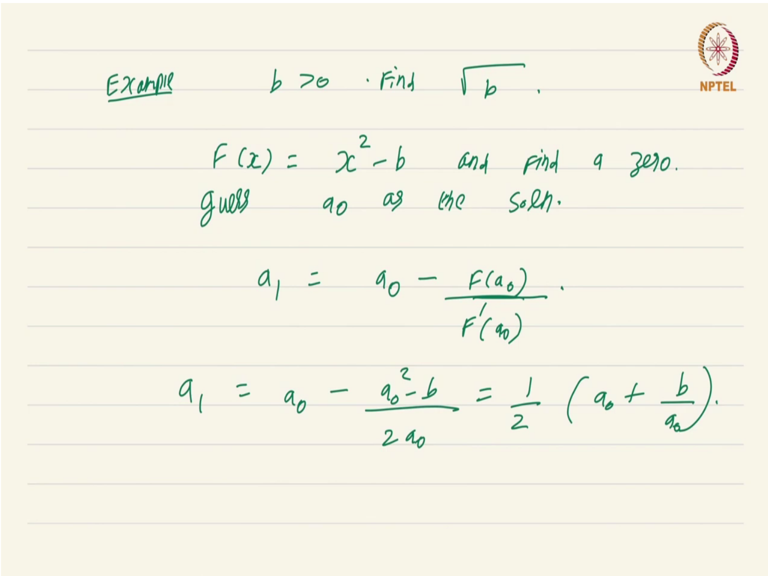

So, the example is as follows. We have b greater than 0, find square root of b, find square root of b, ok. So, what we are going to do is naturally to apply Newton's method. We are going to consider the function F of x equal to x squared minus b and find a 0. This is the aim, ok. So, of course, what does Newton's method say? Guess, a naught as the solution as the solution, ok.

Then, what Newton's method says because we are in the one variable setting, a 1 would be just a naught minus F of a naught divided by F prime of a naught, ok. So, this is just Newton's method in one-dimension. Of course, I am assuming that F prime of a naught is not 0, anyway. For this function the only 0 of the derivative happens at x equal to 0. I am never going to guess 0 as the required square root in any case, ok.

Now, we have the expression F to be x squared minus b. So, if you just substitute everything here you get a 1 is equal to a naught minus a naught squared minus b by 2 a naught which is just half of a naught plus b by a naught. So, this is the divide an average method, ok.

Start with an initial guess, divide b by a naught and take the average and this is the better guess, ok. The reason why this divide and average, this procedure gets you closer and closer to the root is the following intuitive idea. If you had guessed a naught to be very large, then b by a naught will be smaller than square root of b. So, if a naught is very large compared to square root of b, then b by a naught will be smaller than square root of b, and this average will get us closer to square root of b because of that, ok.

So, this is the basic intuition. But at least in this simple scenario we can actually prove that repeatedly iterating this procedure, taking the next guess a 2 to b, half of a 2 plus b by a 2 sorry half of a 1 plus b by a 1 and a 3 to be half of a 2 plus b by a 2, so on and so forth, actually converges to square root of b. We can prove that rigorously. Let us see a proof, ok.

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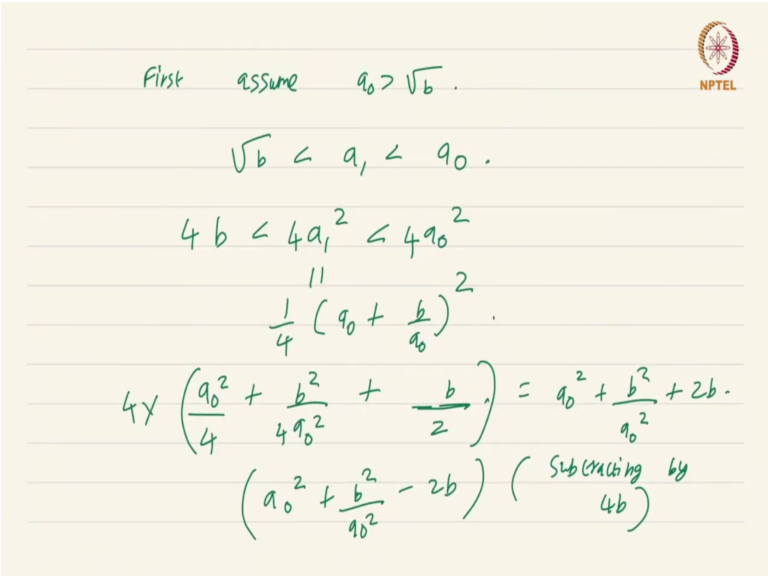

So, we will first deal with the case. First assume that a naught is greater than square root of b, ok. Then, the claim is that square root of b is less than a 1 is less than a naught or in other words that intuition behind this divide an average actually works, ok. Now, showing this is same as showing b is less than a 1 squared is less than a naught square, ok. And here we can substitute for a 1; a 1 is nothing, a 1 squared is nothing but 1 by 4 a naught plus b by a naught squared, ok.

Now, just let us just focus on the middle term. It expands out to a naught squared plus b squared by a naught squared. I should put a 4; put a 4 here plus a naught b by 2. So, this is the sorry not a naught b by 2, just b by 2, that a naught and a naught get cancelled out. So, this is the expansion of the middle term, ok.

Now, what you do is multiply these entire inequalities that we want to show by 4, ok. So, ultimately what you want to show is 4 b is less than 4 a 1 squared is less than 4 a naught square, ok. So, let us take 4 times the middle term, the 4 times the middle term which is nothing but a naught squared plus b squared by a naught squared plus 2 b, ok. So, we want to show that this quantity is greater than 4 b.

So, what we do is we subtract by 4 b, we subtract by 4 b, what you get is a naught squared plus b squared by a naught squared minus 2 b. So, this is after subtracting by 4 b and of course, from elementary algebra this is just a in a naught minus b by a naught the whole squared which we all know is going to be greater than 0, is greater than 0, ok.

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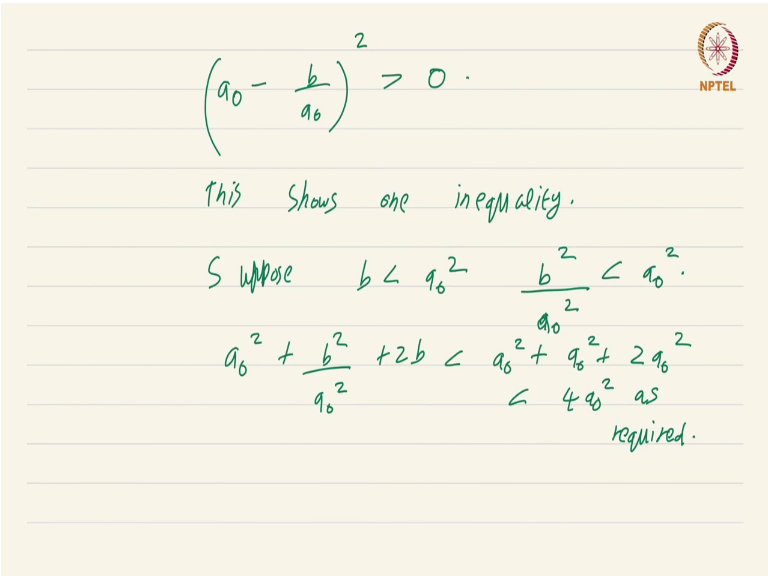

So, this shows the one inequality, this shows one inequality, ok. This shows one inequality. Now, for the other inequality, for the other inequality, suppose we have already b is less than a naught squared, suppose b is less than a naught squared, then b squared by a naught squared, b squared by a naught squared is less than a naught squared.

So, what I have done is I have squared both sides and taken one a naught squared to the other side to get b squared by a naught squared is less than a naught squared, ok. So, this just means that a naught squared plus b squared by a naught squared plus 2 b, this is just the middle term multiplied by 4.

This is less than a naught squared plus a naught squared plus 2 a naught squared, ok. So, I have just used repeatedly the fact that b is less than a naught squared and b squared by a naught squared is less than a naught squared, and this is less than 4 a naught squared as required.

So, if you start with an initial guess a naught, such that a naught squared is strictly greater than b, then it turns out it turns out that the one iteration of Newton's method gives you an a 1 that is strictly greater than a naught, sorry that is strictly greater than square root of b, but is less than a naught. So, a 1 will be greater than square root of b will be less than a naught, ok. So, we have shown this.

But this was under the assumption that a naught is greater than square root of b. How do you get rid of this assumption?

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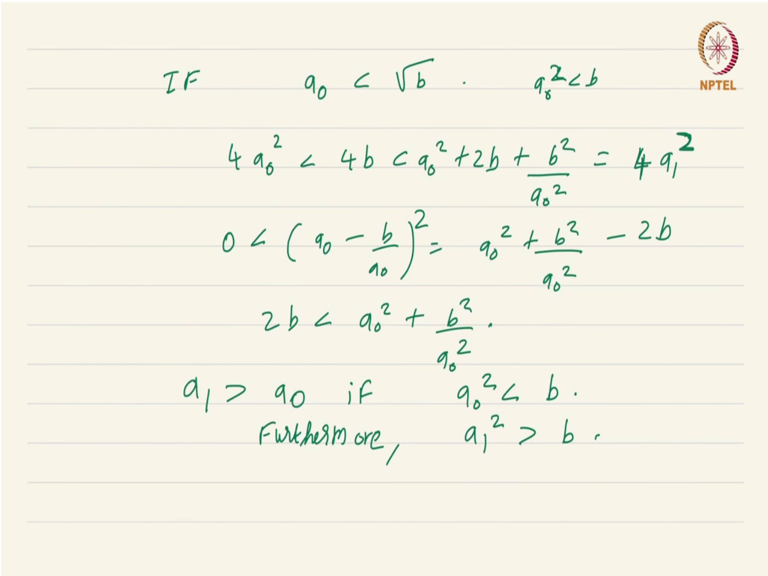

Well, if a naught happens to be less than square root of b if a naught happens to be less than square root of b, then that means, a naught squared is less than b, ok. Then, observe that 4 a naught squared is less than 4 b, which is less than a naught squared plus 2 b plus b squared by a naught squared which is equal to 4 a 1 square, ok. So, how does this follow?

Well, this follows because 0 is less than a naught minus b by a naught squared which is a naught squared plus b squared by a naught squared minus 2 b. And this shows that 2 b is less than a naught squared plus b squared by a naught squared, ok. So, we have plugged this that 2 b is less than a naught squared plus b squared by a naught squared to in this equation here. We have just substituted. We have substituted here, ok.

So, to put this in more concrete terms what happens is a 1 will be greater than a naught, if a naught squared is less than b. Furthermore, a 1 square will be greater than b, ok. So, what happens is once you apply Newton's method with an initial guess less than square root of b, the next, the first iteration, the next value a 1 will be greater than, a 1 will be greater than square root of b, so we move on to the previous case.

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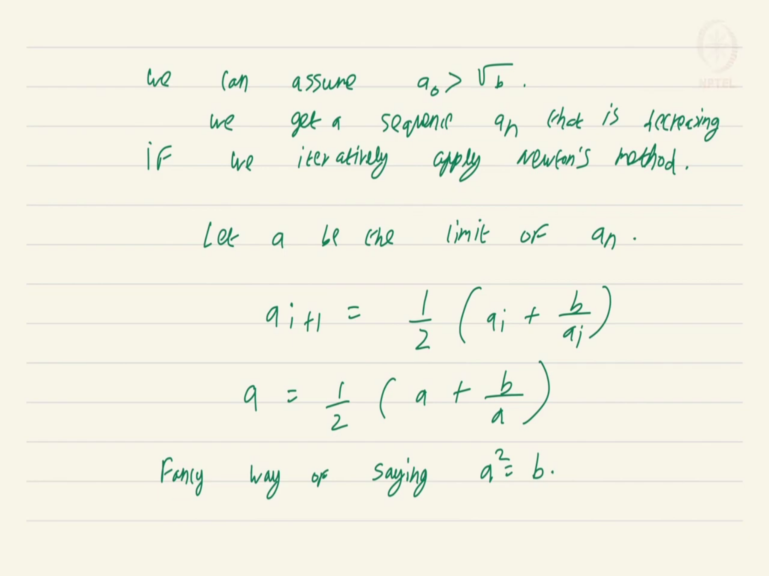

So, we can safely assume we can safely assume our initial guess a naught was greater than square root of b because anyway we will land up in this situation, even if a 1 was less than square root of b. Now, because of this, we get a sequence a n that is decreasing if we iteratively apply iteratively apply Newton's method, ok. So, what we are going to do is start with an initial guess, get an a 1, start that as the new guess, and keep repeating, ok.

Because a decreasing sequence that is bounded below by square root of b has to have a limit, let a be the limit, let a be the limit of a n, ok. Now, at each stage, we had iterated by this procedure a of i plus 1 equal to half of a i plus b by a i, then taking limits on both sides you get a is equal to half of a plus b by a, ok. This is just a fancy way of saying, fancy way of saying a squared equal to b. So, the net upshot is if you keep repeating this divide and average method, you will end up at the required square root of the number b.

So, this was a brief illustration of Newton's method as always Hubbard's book gives this in great detail. Then, the appendix the entire proof of cantor which is theorem is there. And the proof of the inverse function theorem using cantor which is theorem is also present in Hubbard's book. We will use the Banach fixed point theorem to prove the inverse function theorem. However, some parts of the proof will be inspired by Newton's method, though we will not use Newton's method in a direct way.

This is a course on Real Analysis, and you have just watched the video on Newton's Method.