Real Analysis II
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Lecture - 16.1
The Banach Fixed Point Theorem

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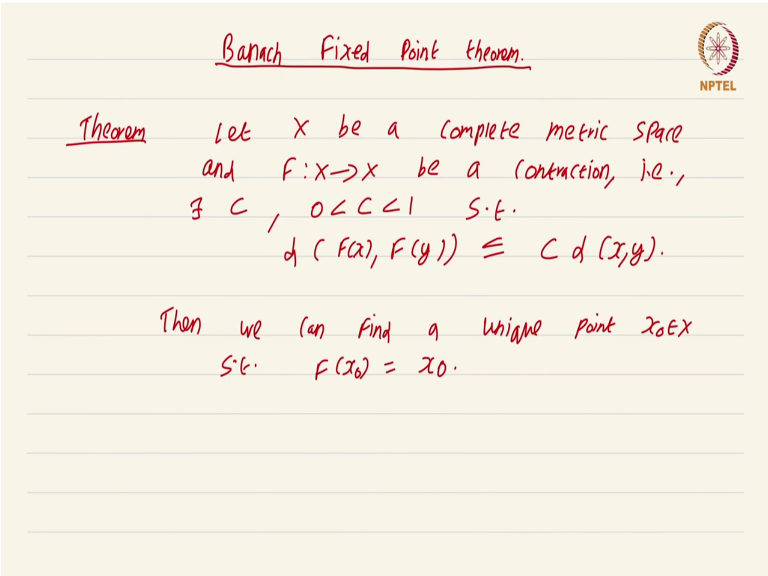

We now move on to the next major theorem that we are going to prove in this course this is the inverse function theorem. This is a very famous theorem and it is extremely useful in various parts of analysis as well as in differential geometry. Now, the classical proofs of the inverse function theorem were quite involved and long, but somewhat elementary. We are now going to use the full force of abstraction to give a very elegant proof of the inverse function theorem.

The proof is still not easy, but it becomes more transparent if we invoke these abstract machinery. To that end let us first prove a fixed point theorem called the Banach Fixed Point

Theorem. This theorem has other uses as well. You can prove the existence and uniqueness of solutions to ordinary differential equations using this theorem and it is used in several places in analysis and functional analysis.

So, this theorem is also known as the contraction mapping principle and you will know in a moment why it is called the contraction mapping principle, let us state the theorem. The setting is a complete metric space. So, let X be a complete metric space and F from X to X be a contraction this just means that there exists a constant C, 0 less than C less than 1 such that d of F of x comma F of y is less than or equal to C times distance of x to y.

So, in some sense the map sort of contracts points x and y to points that are closer to each other and that is captured by saying that d F x, F y is less than or equal to C d x, y. This constant C is independent of the choice of points. So, that is to be remembered. This constant c does not depend on the choice of points.

The conclusion is then we can find we can find a unique point x naught in X x naught in X such that F of x naught is equal to x naught. So, there is a unique fixed point for the mapping F whenever the mapping F is a contraction, the completeness of the space is very crucial which it will become very clear in the methodology of proof why completeness is essential in this result. Let us prove this result.

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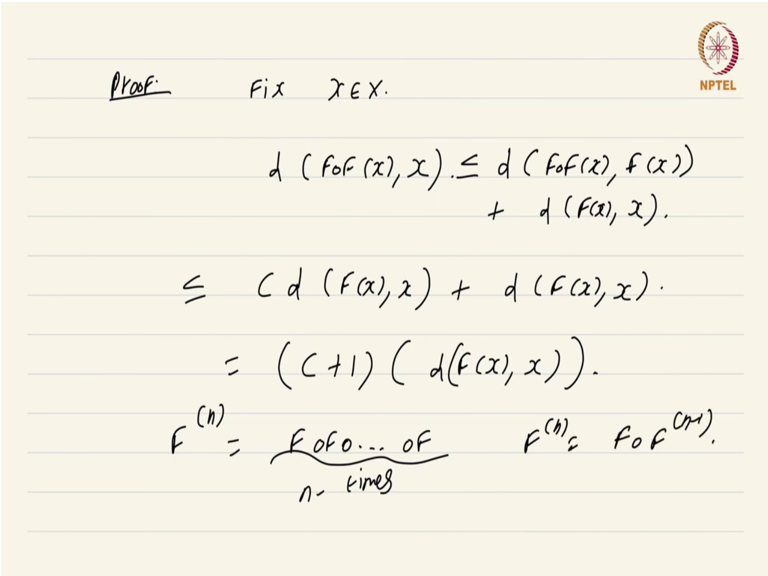

Proof: fix x in X fix a point x in X. Now, let us just see what happens to the point F composed with F of x the distance from F applied to the point x twice in succession let us see what happens to it. Well, by our hypothesis that this is a contraction we know that this is going to be less than or equal to C times d F x F inverse x, but unfortunately F need not be invertible. So, we cannot apply such an argument.

But, what we can do is try to get the required term that we need so, write this as d F compose with F x comma F x plus d F x comma x. So, the reason why I did this is because I need this term to use the hypothesis that F is a contraction. I cannot write this in terms of F inverse of x simply because F; F need not be an invertible map. But, now that we have this we can write the first term the first term simplifies this is less than or equal to C times d F of x comma x plus the same thing again d F of x comma x.

So, the net upshot is this is equal to C plus 1 times d F of x comma x, ok. So, this is the conclusion at the first stage when you apply F to the point x twice in succession we get a nice inequality d F composed with F x comma x is less than or equal to C plus 1 times d F of x comma x. Now, consider the iteration F iterated n times which is just F composed with F composed with F n times, ok or if you want a recursive definition F n is just F compose with F n minus 1 consider the iterates of F.

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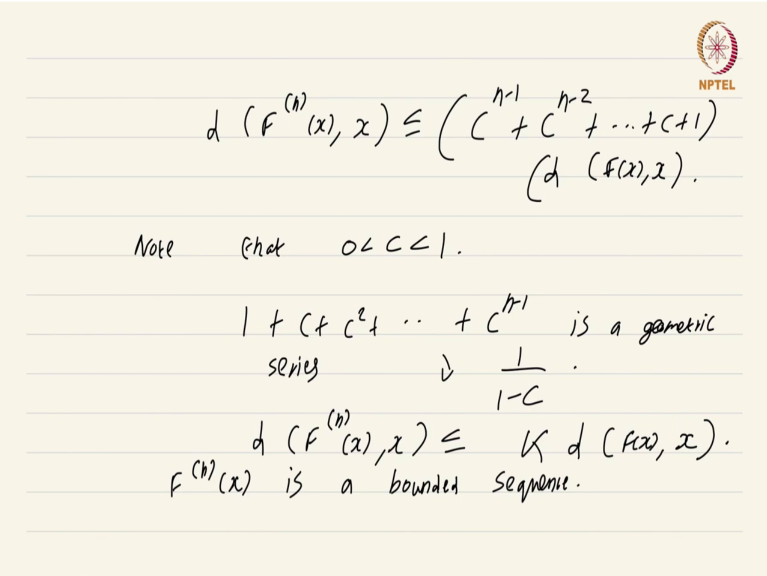

And, let us look at what happens to the point x, d F n x comma x. Now, you can apply a similar start of argument that we have already done and by induction you can conclude that this is less than or equal to C n minus 1 plus C n minus 2 plus dot dot dot plus C plus 1 times d F of x comma x. So, the exact same argument that we applied for the first case for F 2 if you apply induction you will immediately get this ok.

Now, note that 0 is less than C is less than 1 and from this it follows from this it follows that this series 1 plus C plus C squared plus dot dot dot C n minus 1, this is a geometric series. This is a geometric series and this will converge to a quantity when you take limit n going to infinity this will just converge to 1 by 1 minus C, ok. It is a convergent geometric series, ok.

What does this show? This shows that d F n x comma x is less than or equal to some constant I do not know what that is. This is 1 by 1 minus C K d F x comma x, ok and for a fixed x for a fixed x this d F x x is also some constant this is also some constant. Net upshot is this F n x is a bounded sequence, it is a bounded sequence ok. Now, because F n x is a bounded sequence we can try to show that it converges, but we are in an arbitrary metric space so, we have to be a bit more delicate.

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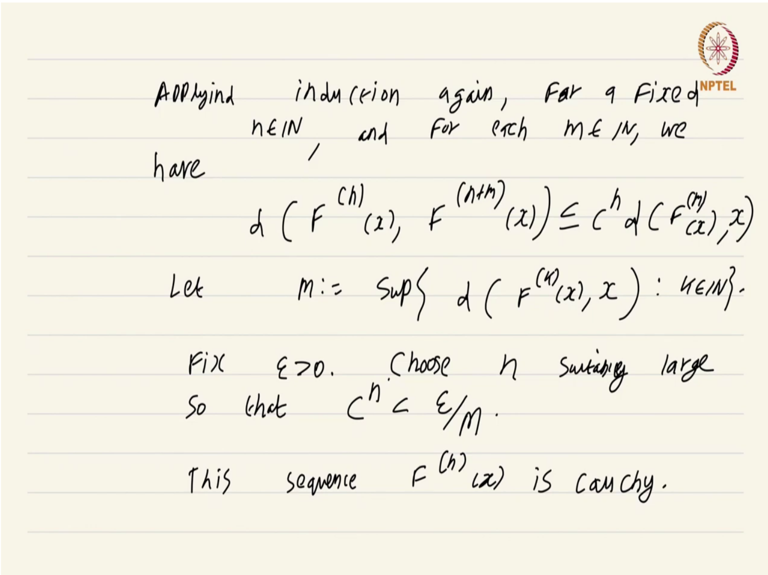

So, we apply induction again applying induction again induction again for a fixed n in N and for each m in N we have d F n x comma F n plus m x F n plus m x is less than or equal to C power n d F n x comma x, ok. So, this prove please take your time and prove this by induction it is very easy and it follows along the same lines of what we have been doing so far, ok.

Now, let m be the supremum of the quantities d F K x comma x as K runs through the natural numbers ok. Fix epsilon greater than 0 fix epsilon greater than 0. So, if n is suitably large so that this C power n is less than epsilon by this supremum which we called m, ok. So, rather than saying if n choose n suitably small will be a better phrasing choose n n suitably large so that C power n is less than epsilon by m ok.

So, what we can conclude from this is this sequence F n x is Cauchy, ok.

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Let x_0 be the limit.

$$F(x_0) = x_0.$$

$$d(F(x_0), x_0) \le d(F(x_0), F^{(n)}(x)) + d(x_0, F^{(n)}(x)) \le$$
$$C\, d(x_0, F^{(n-1)}(x)) + d(x_0, F^{(n)}(x))$$

the RHS can be made small if n is very large. $d(F(x_0), x_0) = 0$.

$$F(x_0) = x_0.$$

So, let x naught be the limit of the sequence. This is the crucial point at which we actually require the completeness of the space. So far in this proof the completeness of the space was completely not needed, ok. We are going to now show that this point is the required is the required fixed point ok.

Now, to see this just observe that d of F of x naught comma x naught is less than or equal to d of F of x naught comma F n of x plus d of x naught comma F n of x ok and this is less than or equal to C times d of x naught comma F n minus 1 of x plus d of x naught comma F n of x ok.

Now, because F n x converges to x naught the RHS can be made small can be made small if n is very large in if n is very large. In other words d F x naught x naught is just 0 ok which just means F of x naught is equal to x naught. So, we have found the required fixed point.

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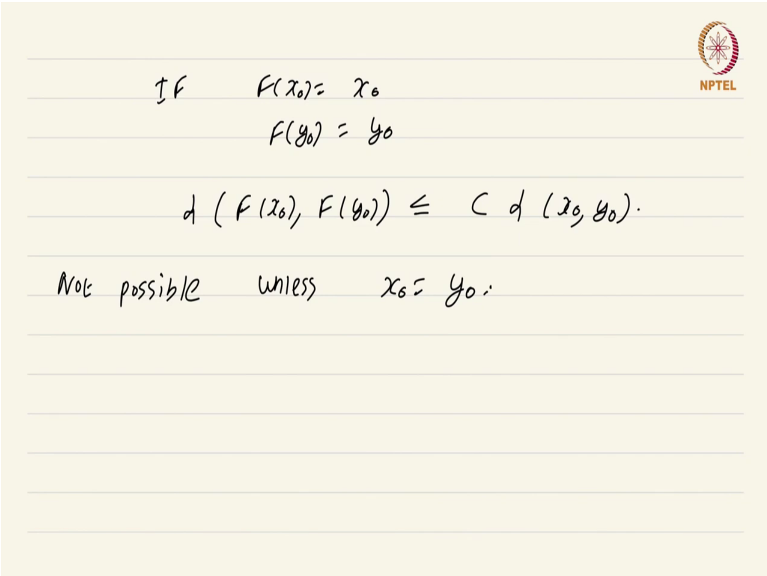

Now, if F of x naught equal to x naught and F of y naught is equal to y naught then d of F of x naught comma F of y naught is less than or equal to C times d of x naught y naught. And, because C is less than one that is simply not possible this is not possible unless x naught is equal to y naught, ok. So, this shows that there can be at the max one fixed point.

We have shown that for a contraction there can be at the max one fixed point here completeness nothing is used; whenever you have a contraction there can be at most one fixed point. So, this concludes the proof of the Banach fixed point theorem. We will soon see how to prove the inverse function theorem using the Banach fixed point theorem.

In the next video I am going to motivate the proof of the inverse function theorem by using another theorem, but that is often used as a substitute for the Banach fixed point theorem Newton's method. This is a course on real analysis and you have just watched the video on the Banach fixed point theorem.