Real Analysis II
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Lecture - 15.2
Taylor's Theorem with Remainder

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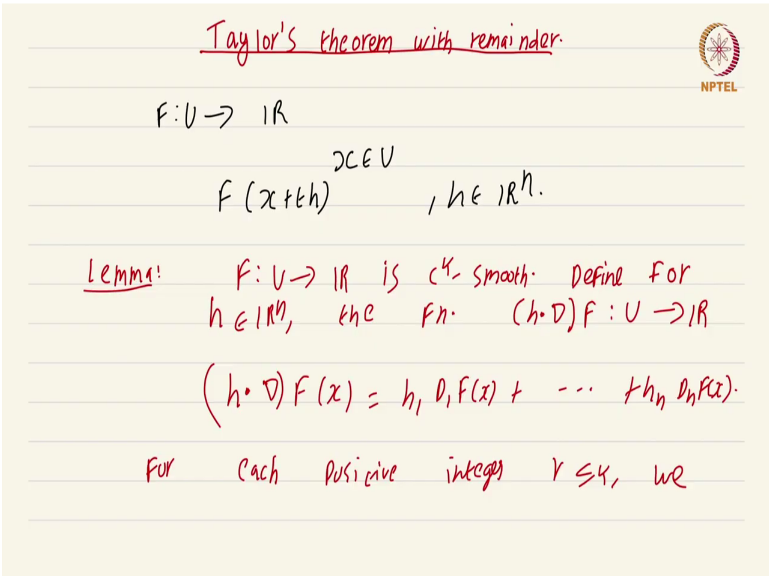

Our existing version of Taylor's Theorem was really elegant and the proof was also very short and cute. Now we want to give a Taylor's theorem with reminder term and the proof is not so elegant. Now, to do this what we are going to do is we are going to reduce a function F from U to R to a one variable function and the way to do that is we are going to fix a point x in U where we are going to write down the Taylor series expansion.

And consider the function F of x plus t h, where h is in R n ok. And now we are going to treat this function as a function of the variable t and apply the classical one variable Taylor's theorem that we are now very familiar with. Now, to do that we have to repeatedly calculate the derivatives of this function the higher order derivatives of this function with respect to t and we need a formula for that.

The next technical looking lemma will take care of this for us the lemma looks complicated because notation has to be set up, but the proof is rather easy. So, F from U to R is C K smooth. We are going to define for h h in R n the object the function h dot delta. So, this is dot h dot delta this is the function from U to R and it is defined by h dot delta of F at the point x is nothing but h 1 D 1 F x plus dot dot dot h n D n F x.

If you carefully observe what we have done is we have treated this dot as the standard dot product and the expression on the right hand side is exactly what you would get. If you blindly just take the dot product between the vector h and the operator nabla treated as a vector ok. So, this is sort of like a formal definition this has no meaning so far ok. For each positive integer r less than or equal to k.

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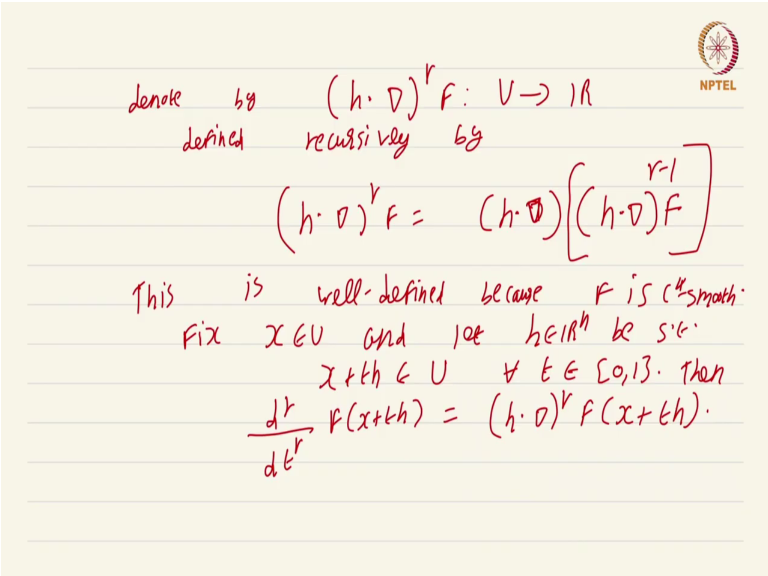

We denote by nabla dot F r F power r sorry h dot nabla power r of F to be the function from U to R defined by defined recursively by; defined recursively by h dot delta r F is equal to h dot delta of h dot not delta I should read it as grad h dot grad h dot grad power r F is equal to h dot grad acting on h dot grad of F h dot grad power r minus 1 of F ok.

So, what is essentially happening is recursively this object h dot grad power r minus 1 acting on F is a function from U to R, then you take h dot grad of that ok. Now you require r to be less than or equal to k, because when you work this out you will notice that derivatives up to order r will occur when you actually sit down and take these derivatives ok. So, this is well defined; this is well defined because F is C k smooth.

Now, fix so this was just set up for this technical lemma fix x in U and let h in R n be such that x plus t h is an element of U for all t belong to close 0, 1 ok. Then d r by d t of F of x plus t h, so taking so this should be d t power r. So, taking the rth derivative with respect to t is nothing but doing h dot grad power r of F of x plus t h ok. So, you can convert the derivative on t to taking derivatives with respect to this new operator h dot grad ok let us prove this.

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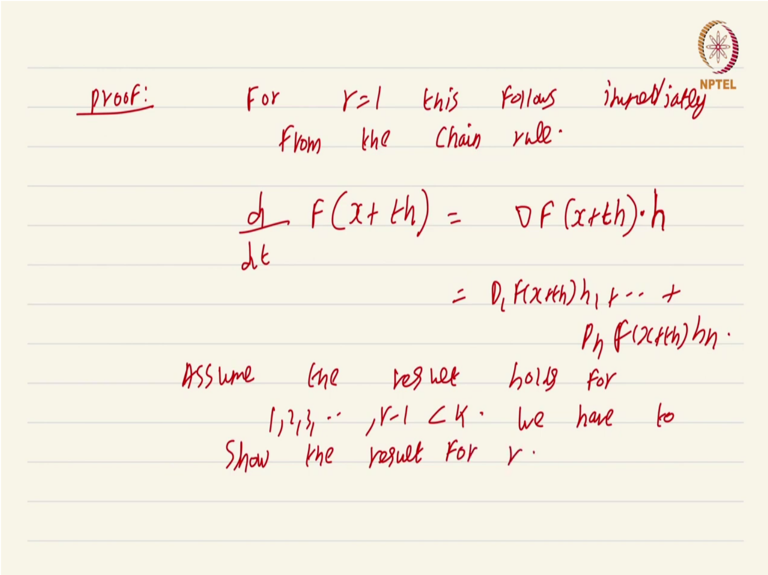

The proof is not hard proof for r equal to 1 this is just R equal to 1 this follows immediately this follows immediately from the chain rule; from the chain rule. Let us see why that is the case we have d by d t of F of x plus t h ok. This is nothing but gradient of F at the point x plus t h at the point x plus t h, then you have to take the derivative of x plus t h with respect to t and that is just going to leave you with the vector h with the vector h ok.

So, this is just i of course, i must put a dot product and this is nothing but D 1 F of x plus t h h 1 plus dot dot dot D n F of x plus t h h n. So, the base case of the induction is straight forward. Now, assume the result holds for 1 2 3 dot dot dot r minus 1 all of these are less than k ok. Now we have to show the result for r we have to show the result show the result for r fine.

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$$\frac{d^r}{dt^r} F(x+th) = \frac{d}{dt}\left(\frac{d^{r-1}}{dt^{r-1}} F(x+th)\right)$$

$$= \frac{d}{dt}\left((h\cdot\nabla)^{r-1} F(x+th)\right)$$

$$= (h\cdot\nabla)^r F(x+th).$$

So that means we have to compute; we have to compute d r by d t power r F of x plus t h which we know is nothing but d by d t of d r minus 1 by d t power r minus 1 god I hate this notation for higher derivatives F of x plus t h ok. Which we know by induction hypothesis is nothing but d by d t of h dot grad power r minus 1 F of x plus t h ok and this is just by the definition of h dot grad and the fact that the result holds for the one variable case this is just h dot grad power r F of x plus t h.

So, the proof was just essentially symbol pushing where we have just used the induction hypothesis as well as the base case of the induction ok. Now in one of the exercises you are asked to expand out what this h dot grad power r is going to look like in terms of the higher order partial derivatives so please do that.

Now, proving Taylor's theorem is a piece of cake all the hard work has been pushed inside this lemma I will just prove one version leaving the integral form of the reminder to you Taylor's theorem with reminder.

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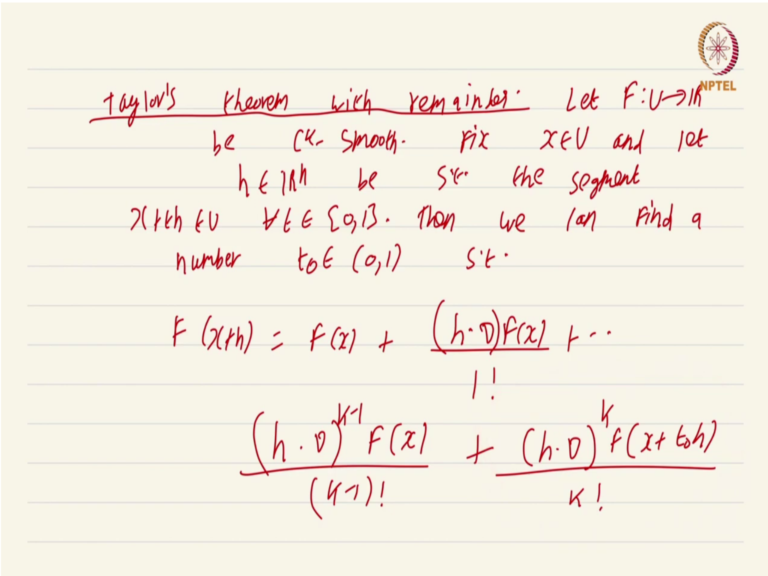

So, this approach to Taylor's theorem is notationally not pleasing, but it illustrates the value of what I like to call the teakettle principle that is the I am not going to elaborate with the story. Please search Teakettle principle on Google to get a story of why this is going to be called the Teakettle principle, the basic idea is if you are confronted with a complicated problem reduce it to a simpler problem.

We have already solved Taylor's theorem in great detail multiple forms of the remainder term and all that in one variable reuse that technology, so that is what we are going to do.

Let F from U to R be C k smooth; be C k smooth fix x in U; fix x in U and let h in R n be such that the segment the segment x plus t h is in U for all t in 0, 1 ok. Then we can find; we can find a number t naught in 0, 1 such that F of x plus h is nothing but F of x plus h dot grad F of x by 1 factorial.

So, technically I should write it like this h dot grad F x at h dot grad F at the point x 1 factorial plus dot dot dot h dot grad power k minus 1 F at the point x plus no at the point x at the point x by k minus 1 factorial plus the final reminder term which is what we were interested in this is nothing but h dot grad power k of F of x plus t naught h divided by k factorial.

So, this is going to be the remainder term this is the remainder term ok. So, the statement looks a bit complicated, but the proof is just reduction to the one variable case.

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Proof: $g : (-\epsilon, 1) \to \mathbb{R}$ defined

$t \mapsto F(x+th)$ for some suitable

by Taylor's thm. one variable case $\epsilon > 0$

$$g(1) = g(0) + \frac{g'(0)}{1!} + \cdots + \frac{g^{(k-1)}}{(k-1)!} + \frac{g^{(k)}(t_0)}{k!}$$

$t_0 \in (0,1)$. The previous lemma finishes the proof.

So, how does the proof go well consider the function g from minus epsilon 1 to R defined by t goes to F of x plus t h. This is well defined because the entire segment from x to x plus h is fully contained in U and U is an open set, because U is an open set I can sort of push this a little bit further and take it to minus epsilon. So, I must write for some some suitable epsilon greater than 0 ok

So, if you choose epsilon small enough this is actually well defined from minus epsilon all the way to one then usual Taylor's theorem gives g of 1. So, by Taylor's theorem Taylor's theorem one variable case one variable case we get g of 1 is nothing but g of 0 plus g prime 0 by 1 factorial plus dot dot dot g power k minus 1 by k minus 1 factorial plus g the kth derivative at t naught by k factorial where t naught lies in 0, 1. This is just the classical Taylor's theorem.

Now, the result is exactly done the previous lemma previous lemma finishes the proof, I am not going to elaborate anymore its rather easy previously finishes the proof. So, we just reduced everything to the one variable case and we have got a formula for Taylor's theorem. Admittedly this is not as elegant a formula as what we got before for the Taylor polynomials, but that version of Taylor's theorem has the deficiency that you only know that the remainder term is rather small, but you do not actually know it explicitly.

This is a course on real analysis and you have just watched the video on Taylor's theorem with reminder.