Real Analysis II
Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 14.1
Higher-Order Derivatives

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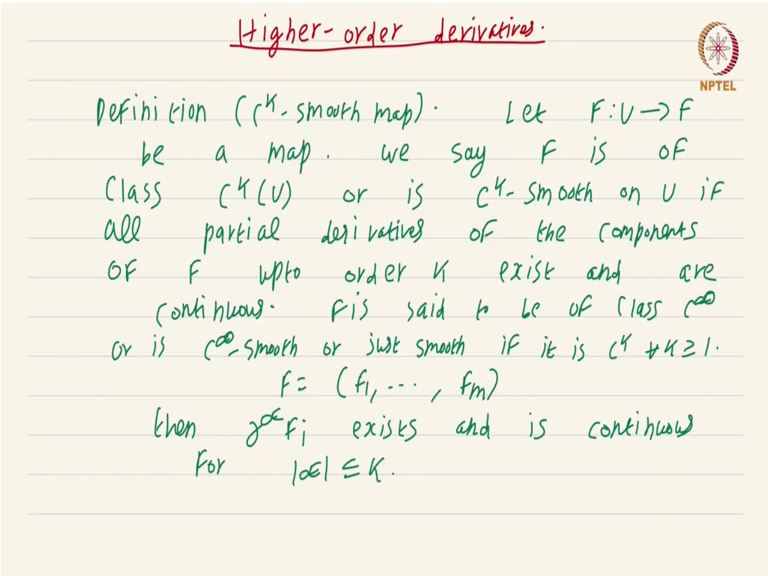

In this video we are going to discuss higher order derivatives. We have already seen what it means for a scalar valued function of a vector variable to be C K smooth, we make the analogous definition for a general map between Euclidean spaces the definition is very simple definition this is the definition of a C K map C K smooth map.

So, as usual let F from U to F be a map as always U is a subset of open subset of E. We say F is of class C K of U or is C K smooth on U, if all partial derivatives all partial derivatives of the components of F; components of F; components of F up to order K exist and are continuous. So, in other words if you just write F as F 1 dot dot dot F m.

Then del alpha F i exists and is continuous; and is continuous for mod alpha less than or equal to K ok. So, all the partial derivatives up to order K of the coordinate functions or the component functions of F F 1 to F m are exist and are continuous. Now this is a perfectly reasonable definition its straightforward and its exactly the same as what is there for a scalar valued function of a vector variable.

However, if you go back to the definition of the derivative in this general setting of a map F from U to F there was no coordinates involved in the definition. The definition just use the fact that the space E and F were norm vector spaces we did not use anything more than that.

So, artificially introducing coordinates in this definition though perfectly reasonable is not the most elegant way, is there a way to define the notion of a C k smooth map in general without resorting to going to these coordinate functions F 1 to F m is there some sort of intrinsic way to do this. Intrinsic is a word, that means that uses only the structure of the given space and not anything more.

Something that does not like rely on something like coordinates essentially what are F 1 to F m, these F 1 to F m come from the fact that you choose a basis for the vector space F. And in terms of the basis you can write the function F as F 1 E 1 plus F 2 E 2 plus dot dot F m E m this is what is a non intrinsic or an extrinsic way of defining the thing. Now we want to see whether we whether it is possible to define it in an intrinsic way.

So, this is going to be a bit abstract and at first it might not be clear how exactly the things are going to work. So, we will first deal with the C 1 case and then let us just define what it means for a function to be C 1 without resorting to coordinates and then see the general case.

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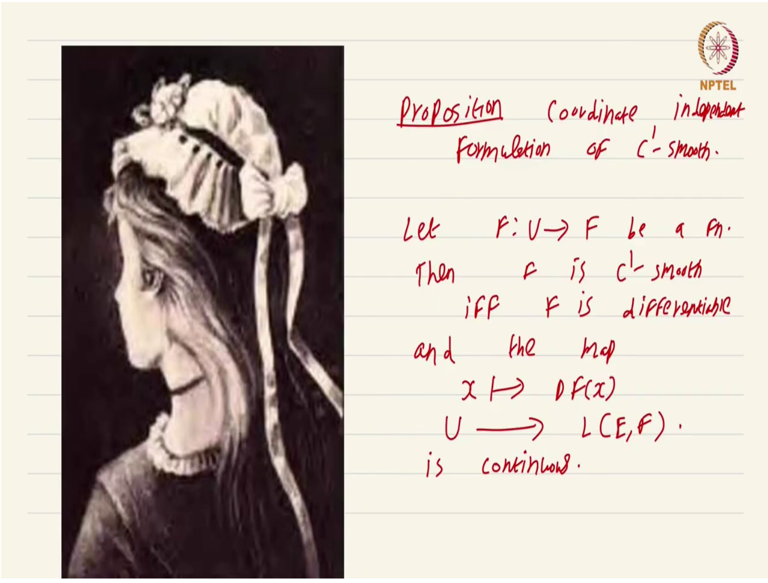

So, as a interesting aside let me show you this famous picture this is a picture from a 1888 postcard from Germany, this is a famous picture you have probably seen this before depending on the perspective you look at you either see an old woman’s side portrait or a very young girls turning back turning her head and neck towards the side.

So, depending on your perspective you see 2 different things and suddenly your viewpoint shifts. So, such a shift is going to happen now at first it might be difficult there might be no easy way to see the perspective changing, but once it clicks it will be very difficult to go back to the previous picture; you cannot un see that this single picture represents both a young lady as well as an old woman ok.

So, that was just interesting aside lets write down the proposition for the C 1 case and prove it then move on to the general setting which is a bit abstract ok.

So, this is the coordinate independent formulation of C 1 smoothness of being C 1 smooth. Of course, there is one thing that I forgot to mention in this definition let me just fix it right away F is said to be of class C infinity or a C infinity smooth or just smooth without any further adjectives if it is C K for all K greater than or equal to 1.

In other words this is just a somewhat opaque way of saying that all partial derivatives of all orders of the component or coordinate functions exist and are continuous. The continuity is. In fact, redundant as you have seen when we dealt with scalar valued functions of a vector variable ok.

Coming back to this coordinate independent formulation of C 1 smooth, let F from U to F be a function then F is C 1 smooth if and only if; if and only if F is differentiable; F is differentiable and the map x going to DF x this is a map that starts in U and maps to L EF. So, think about this for a moment each DF x is a linear map from E to F. So, if you consider the map that takes a given point x and maps it to this entire linear map DF x you land inside the space L EF ok.

So, the map x going to DF x is continuous, of course if nothing is said the topology or rather the metric on L EF is the metric coming from the operator norm ok. Always we will put the operator norm on L EF. So, being C 1 smooth is exactly the same as derivative existing and this map x going to DF x being continuous ok.

Let us prove this; let us prove this. So, I am going to make a slight change to the hypothesis I am going to add what I am going to do is I want to concentrate on the smoothness and not exactly on the derivative existing. So, what I am going to do is I am going to slightly change the statement slightly change this function I mean change the hypothesis.

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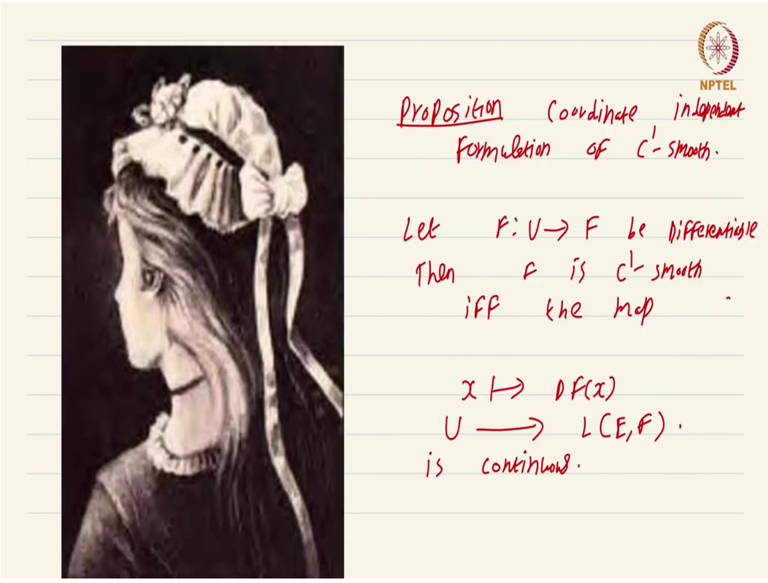

So, what I am going to do is let F from U to F be differentiable then F is C 1 smooth if and only if the map x going to DF x is continuous. So, I am focusing on the C 1s part not on the differentiability. So, I am shifting over the differentiability to the hypothesis ok this is a minor change its not that important, but it sort of clarifies what I want to capture.

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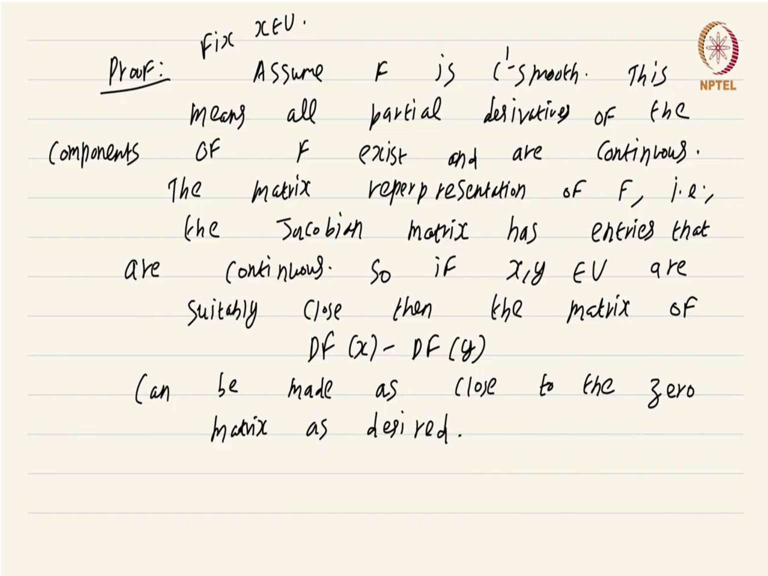

Let us prove this the proof is not hard and but its interesting proof. Now first assume F is C 1 smooth assume F is C 1 smooth, this means all partial derivatives of F exist not of F of the components; of the components of F exist and are continuous; and are continuous ok. Now this just means that the matrix representation of F that is the Jacobian matrix; the Jacobian matrix has entries the entries are precisely the partial derivatives and entries that are continuous; entries that are continuous ok.

So, if x comma y in U are suitably close are suitably close, then the matrix of DF x minus DF y can be made as close to the zero matrix as desired. I am being a bit loose here hoping that you will be able to make this precise its rather easy to make what I am saying precise.

So, since the partial derivatives are continuous and the fact that the derivative map the matrix representation is going to be just the partial derivatives. If you consider the matrix representation of DF x minus DF y that is going to be very very close to the zero matrix's, if I choose x and y suitably close ok. So of course, I must write fix x in U we fix x in U.

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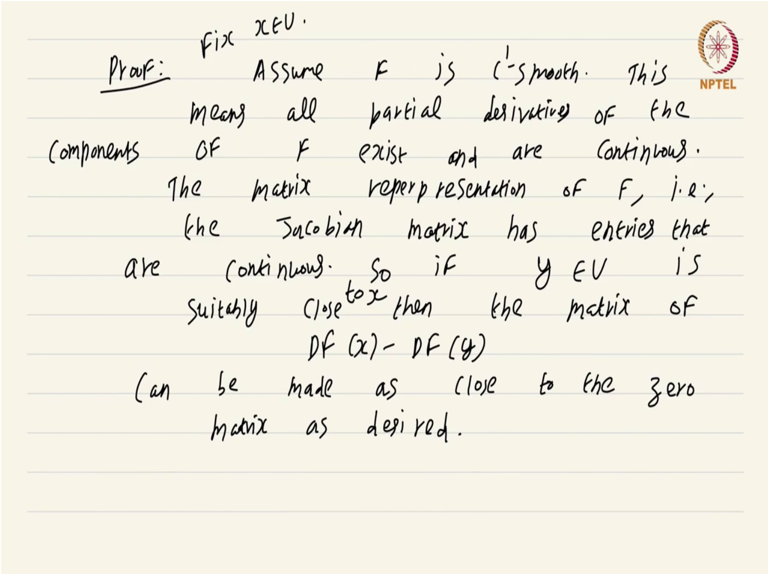

And here I must say if y in U is suitably close to x that is the precise statement. If y in U is suitably close to x then the matrix of DF x minus DF y is going to be the each entry can be made as close to 0 as you desire.

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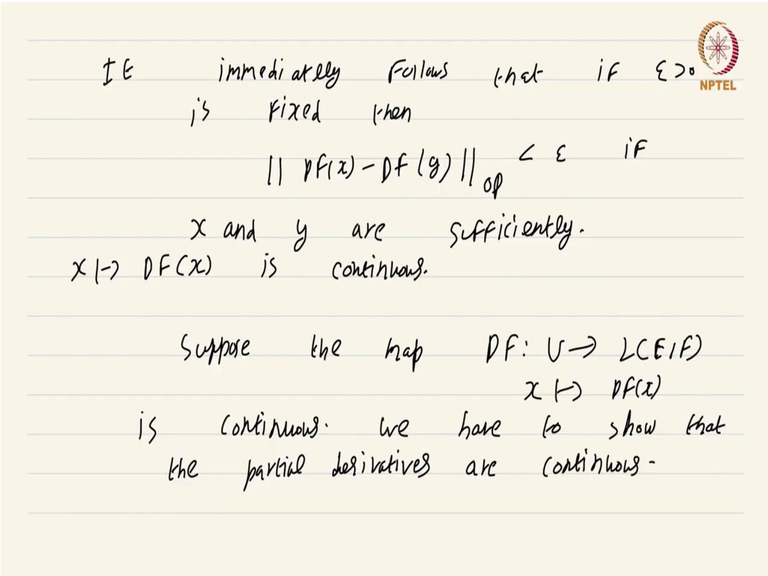

It immediately follows; it immediately follows that if epsilon greater than 0 is fixed, then the operator norm of DF x minus DF y is less than epsilon if x and y are close are sufficiently close. This just follows because we have already established an inequality that involves the operator norm and the various entries of the matrix it was an exercise to relate the operator norm and the various entries of the matrix. If the entries of the matrix are all very very close to 0, then the operator norm can be made really small ok.

So, in other words d the map x going to DF x is continuous ok. So, this takes care of one direction now suppose the map I am going to call this the map DF map DF from U to L EF x going to DF x, suppose this map is continuous we have to show; we have to show that the partial derivatives; the partial derivatives are continuous ok.

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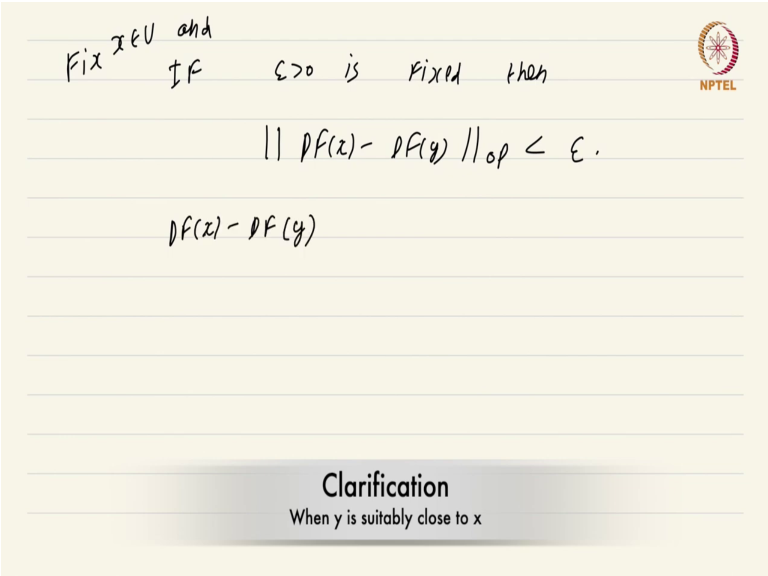

Now, observe that what this means is that if epsilon greater than 0 is fixed, when we operator a norm of course a fix x also fix x in U and if epsilon greater than 0 is fixed then the operator norm of DF x minus DF y is less than epsilon this is just the meaning of the map DF is continuous ok.

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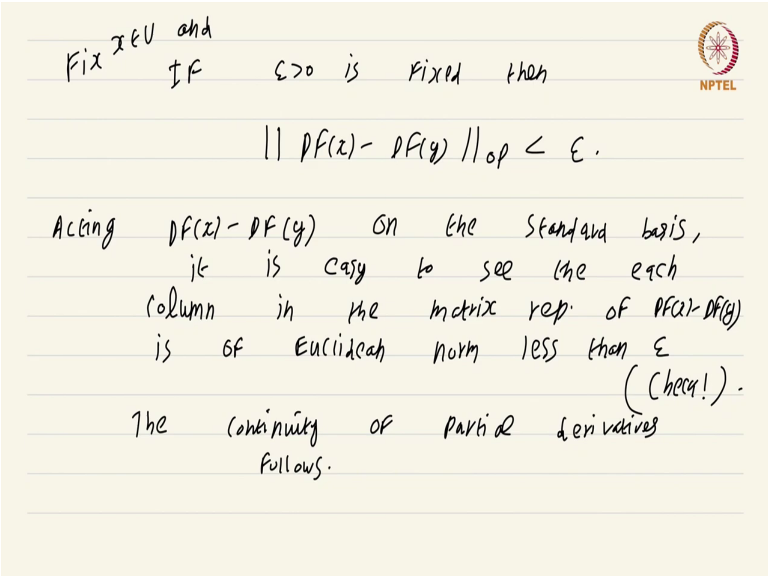

So, if you act DF x minus DF y if you act DF x minus DF y. So, acting on the standard basis; on the standard basis; on the standard basis, it is easy to see; it is easy to see easy to see that each column in the matrix; in the matrix representation of DF x minus DF y. So, it is easy to see that each column in the is of Euclidean norm less than epsilon check this check.

So, what I am doing is I am acting DF x minus DF y on the standard basis and I am going to look at the magnitude of the resultant vector in the Euclidean norm it will be less than epsilon. This just means that when you consider the matrix representation of DF x minus DF y each entry is going to look really small, because each column when you treat it as a vector its going to have magnitude less than epsilon either Euclidean norm less than epsilon.

So, from this the continuity of partial derivatives is immediate continuity of partial derivatives is immediate partial derivatives follows. Of course, I am repeatedly using the fact that if a function is differentiable then the matrix of partial derivatives is the matrix representation of the derivative map ok. Excellent we have got a coordinate independent formulation of the notion of C 1 smoothness, if you observe we did use the coordinate representation in the proof.

We have use the coordinate representation in the proof. However, we have gotten rid of the coordinate dependence in the definition ok.

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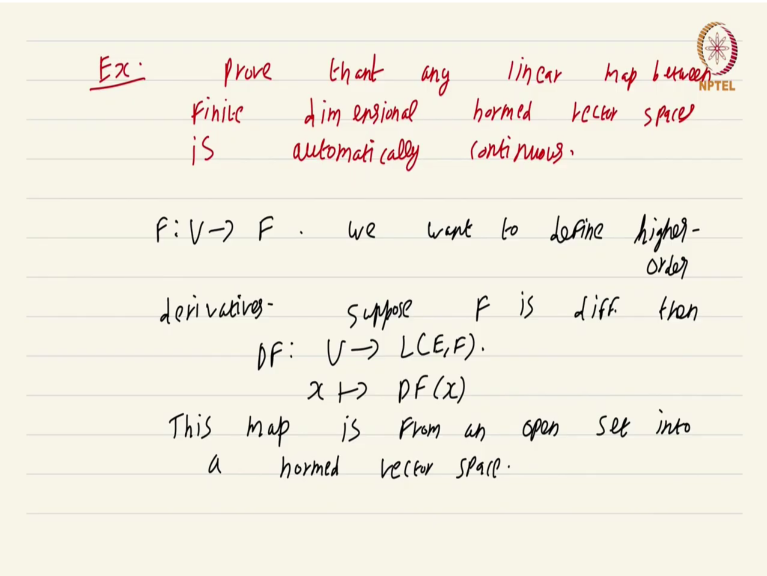

I am going to give you an exercise which should be very very familiar to you by now you must have solved variance of this before prove that any linear map between finite dimensional norm vector spaces; norm vector spaces is automatically continuous ok. So, once you have done this exercise what follows will start to make sense. So, the setup is you are given a map F from U to F we want to define higher order derivatives we want to define higher order derivatives; higher order derivatives.

So, suppose F is differentiable; suppose F is differentiable, then as we have seen we get a map DF from U to F we just takes an element x to the corresponding linear map DF x ok. So, actually this is not DF from U to F this is DF from U to L EF ok. So, you get a map that takes a vector x and maps it to the linear map DF x which is an element of L EF. Now this map is from an open set into a normed vector space right a normed vector space L EF.

Now, at this point there are two ways to proceed the first is to observe that L EF can be treated as a space of matrices and as we have done before we can identify matrices with Euclidean space and then define the higher order derivatives using the fact that this space is also an Euclidean space. We treat this map DF as a map from an open set in an Euclidean space mapping into an Euclidean space and you can define the derivative as usual.

But that again is not so elegant and its somewhat coordinate dependent, there is a coordinate independent way of defining what it means for this map DF to be differentiable.

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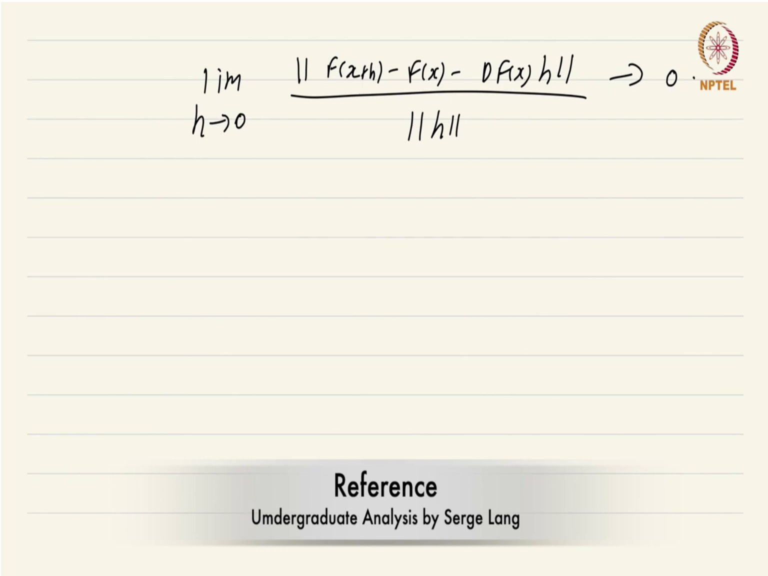

And that is the observation that when we define the notion of derivative recall we one way of defining the derivative will be to take limit h going to 0, norm of F of x plus h minus F x minus DF x h by norm h this goes to 0. Observe that in this definition there is no Euclidean structure at all, all it relies on is the fact that both the domain and the codomain are non-vector spaces. In the numerator we are taking the norm in the codomain in the denominator we are taking the norm in the domain.

So, really what is being used is the norm vector space structure and the structure the Euclidean structure is not at all needed at least to make sense of this definition. Except one change in general if you have a linear map DF x between infinite dimensional vector space that it need not be continuous, linearity does not automatically mean continuity unless you happen to be in the finite dimensional case. Now the previous exercise will start to make sense why did we put that exercise.

So, this very same definition would have worked even for infinite dimensional spaces. Of course you have to take DF x is linear you have to put that as an additional sorry you have to put take DF x is not only linear, you have to take DF x is continuous when you are dealing in infinite dimensional spaces that is not automatic. However, we are not going to take the most general case I am going to give references to textbooks by Deodon and Lang.

These textbooks have this in great detail, we are going to just treat the second derivative case in the Euclidean setting that is a somewhat concrete case and even here it is a bit abstract. A I will leave it to you to take care of the general case on your own in your own free time you can read up either Lang or Deodinous book or even Kartans book to see this general setting.

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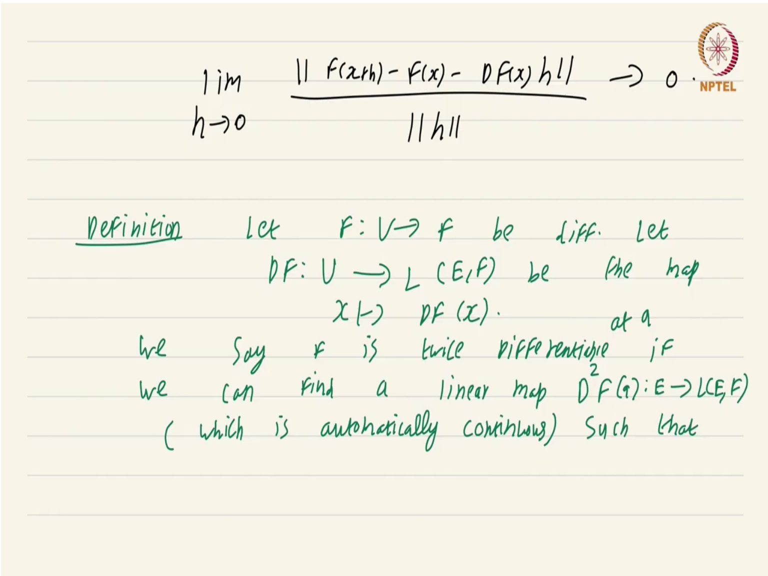

So, let us define the notion of second derivative definition let F from U to F be differentiable. Let DF from U to L EF be the map; be the map x going to DF x as usual, we say F is twice differentiable; is twice differentiable with ok. We say F is twice differentiable if we can find; if we can find a linear map a linear map which we are going to denote as D 2 F to denote the second derivative.

We can find a map I must say which point F is twice differentiable at a, if we can find a linear map D 2 F a from this open set U sorry from E the co domain is what is complicated its L EF, we can find a linear map from E to L EF and this linear map is automatically continuous why because we are in the Euclidean setting. So, everything is finite dimensional including the space L EF which is automatically continuous such that and this is the crucial thing, so I am going to shift over to new page.

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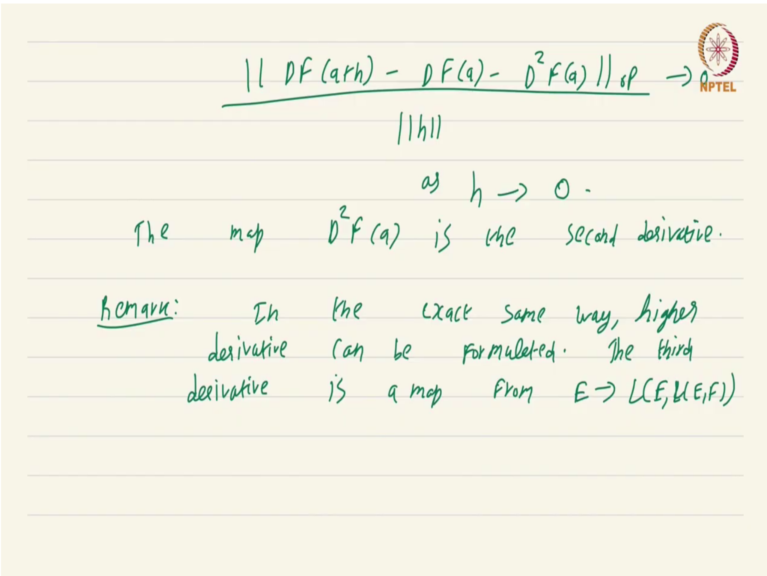

Norm of DF a plus h minus DF a minus D 2 F a the whole things operator norm divided by norm h divided by norm h goes to 0 as h goes to 0 ok. The map D 2 F a is the second derivative; is the second derivative. So, this is a coordinate independent formulation of the second derivative.

So, let me make a remark here in the exact same way; in the exact same way higher derivatives can be formulated; can be formulated. So, for instance the third derivative; the third derivative is a map from E to this very complicated space L E of L EF ok. please ponder over this why do you why do we get L E of L EF ponder over this and think about the higher derivatives I am not going to pursue this any further, I am going to leave it to you to work it out yourself or refer to the textbooks that I have talked about. Now this is very abstract and somewhat complicated.

Let us try to understand this from a different angle when you approach an abstract object through different angles hopefully it starts to become concrete. So, what we are going to do is we are going to analyze what happens.

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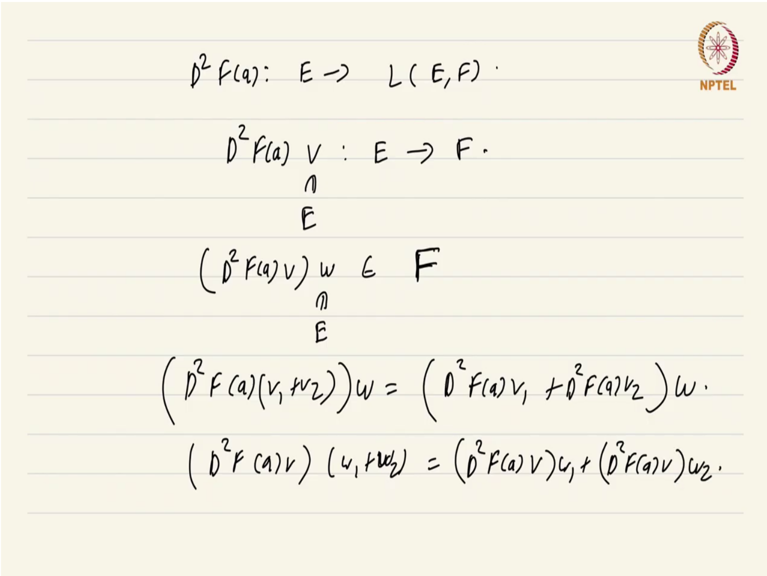

So, this D 2 F a is a linear map from E L EF. So, what does it feed on it feeds on a vector from E ok. So, if you start with the vector from E it feeds on it and it produces for you a linear map from E to F ok. So, that is what this does it eats up vectors in E and produces linear maps from E to F. Well a linear map from E to F also eats vectors, so if you take this object which is a linear map and act it on a vector w from E again ok. What do you get? You end up with an element of F right because L EF is a linear map from E to F.

So, you can think of this D 2 F a as eating two vectors from E v and w and producing a vector F in F. This should start to ring bells and light bulbs should go above your head. What does the linearity mean well linearity in the variable v just means that D 2 F a v 1 plus v 2 acting on w is just D 2 F a v 1 plus D 2 F a v 2 this whole thing acting on w this is just what linearity means.

In a similar way D 2 F a v w w 1 plus w 2 this is using the fact that D 2 F a v is a linear map in L EF this is just D 2 F a v w 1 plus D 2 F a v w 2 something nice has happened.

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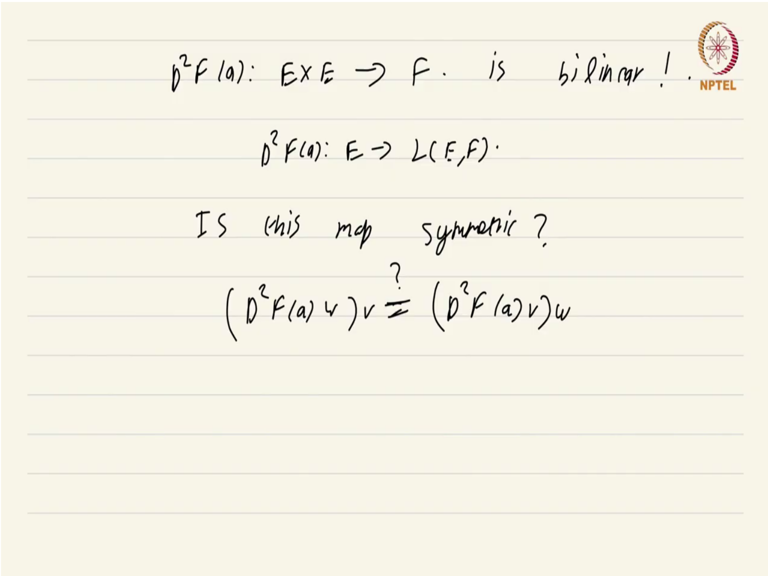

This just says that D 2 F a is a bilinear map from E cross e to F right that is another way of looking at it. Instead of treating D 2 F a as a linear map from E to L EF we can also view it as a linear map from E cross E to F ok. So, already a lot of ideas have been exposed in this particular video, in the next video I am going to analyze when this map is going to be symmetric is this map symmetric.

What is the meaning of is this map symmetric does it follow that if I treat D 2 F a w v is this the same as D 2 F a v w and this should be very familiar to you because we already proved something similar to this.

We saw that if a map F is going to be C 2 smooth at least a scalar valued function sorry. If you take a scalar valid valued function of a vector variable and if its C 2 smooth. Then the mixed partial derivatives are equal do mixed partial derivatives come into the picture. Well we will see it in the next video where we talk about symmetry of the second derivative and also introduce the hessian matrix which is a concrete way of seeing what this bilinear map is going to be.

So, this video is a bit abstract and a bit difficult. So, kindly watch this video again before moving on to the symmetry of the second derivative and the hessian matrix. This is a course on Real Analysis and you have just watched the video on higher order derivatives.