Real Analysis II
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Lecture - 13.3
Differentiation under the Integral Sign

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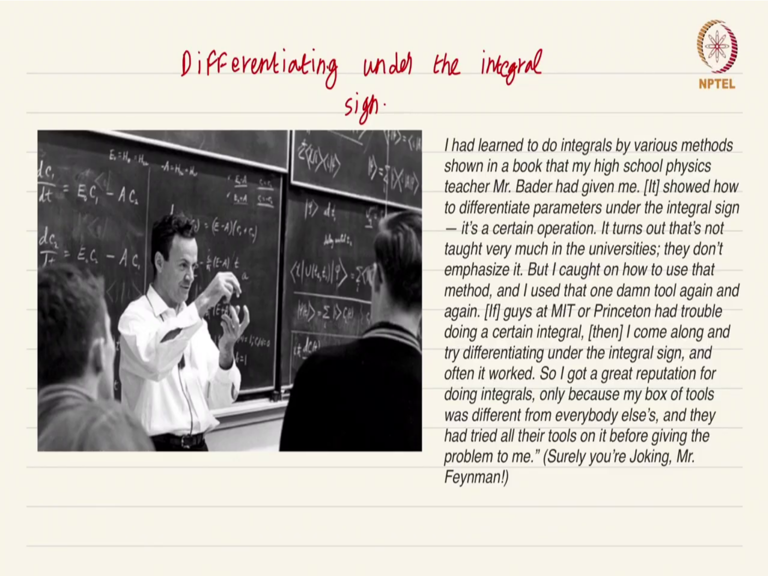

Here you see a picture of the Nobel Prize winner and one of the most famous expositors of physics Richard Feynman, Richard Feynman had a reputation of being able to do the toughest of integrals with ease. Here is a passage from a semi biographical book surely you are joking Mr Feynman. I had learned to do integrals by various methods shown in a book that my high school physics teacher Mr Bader had given me.

It showed how to differentiate parameters under the integral sign it is a certain operation, it turns out that is not taught very much in the universities they do not emphasize it. But I caught on how to use that method and I used that one damn tool again and again if guys at

MIT or Princeton had trouble doing a certain integral then I come along and try differentiating under the integral sign and often it worked.

So, I got a great reputation for doing integrals, only because my box of tools was different from everybody else and they had tried all their tools on it before giving the problem to me. So, this is a very very practically useful tool that will allow you to solve difficult integrals, the importance of this tool is nowadays not that much simply because we have symbolic packages like wolfram alpha that can do differentiation I mean that can do integration. But nevertheless it is a tool that is good to know.

So, I am going to state and prove one version of differentiating under the integral sign, this will turn out to be very useful when you do stuff like complex analysis and you want to prove some integral formulas you can just differentiate under the integral sign. It will allow you to give very elegant proofs of certain theorems, so without much further ado the statement of the theorem.

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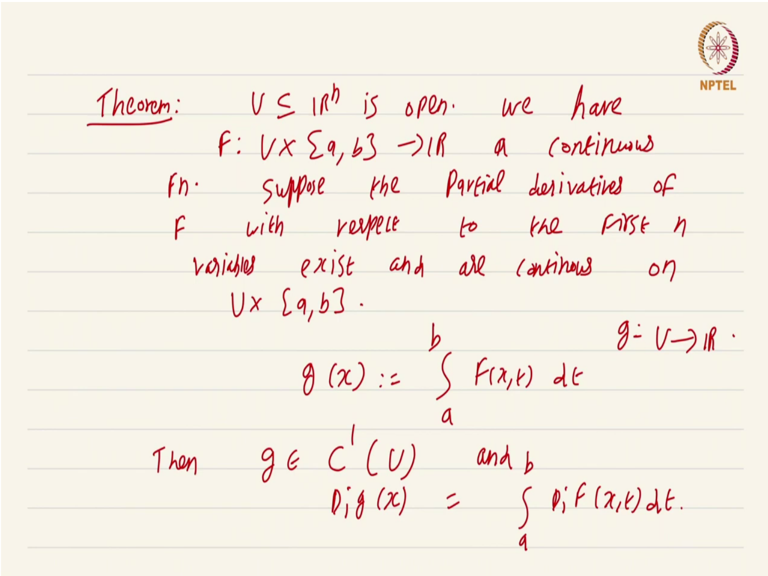

Theorem the setup is as follows U subset of R n is open and we are given we have F from U times the close interval a b to R a continuous function ok. And note think carefully what this means continuity is going to be defined using some metric on the product here it is going to be the Euclidean metric at a continuous function ok.

Suppose the partial derivatives the partial derivatives of F with respect with respect to the first n variables first n variables exist and are continuous and are continuous on U cross a b ok. So, this is essentially just saying that the function is differentiable partial at least the functions partial derivatives exist with respect to the variables coming from U. So, in some sense this function is differentiable when treated as a function of U ok.

So, each partial derivative will define a function from U cross a b to R that function should also be continuous. Now what we are going to do is we are going to define this new function g of x which is given by integral a to b of F of x t d t. So, in a sense g is a function from U to R.

So, for a fixed x you fix the variable slot x here in the function F you get a function of t F of x t, this function is obviously going to be continuous from the facts we have established when we studied metric spaces. Then just integrate this function with respect to t that will give you the function g of x that will give you the value g of x in this way you get a function g from U to R.

Now, what differentiating under the integral sign says is then g is actually a C 1 function on U it is continuously differentiable partial derivatives exist and are continuous. And D i the partial derivatives D i of g of x is nothing but integral a to b D i of F of x t d t ok. So, the partial derivatives exist and are continuous and the partial derivatives can be obtained by simply shifting the derivative operator D i inside the integral sign ok. Let us prove this the proof is not hard.

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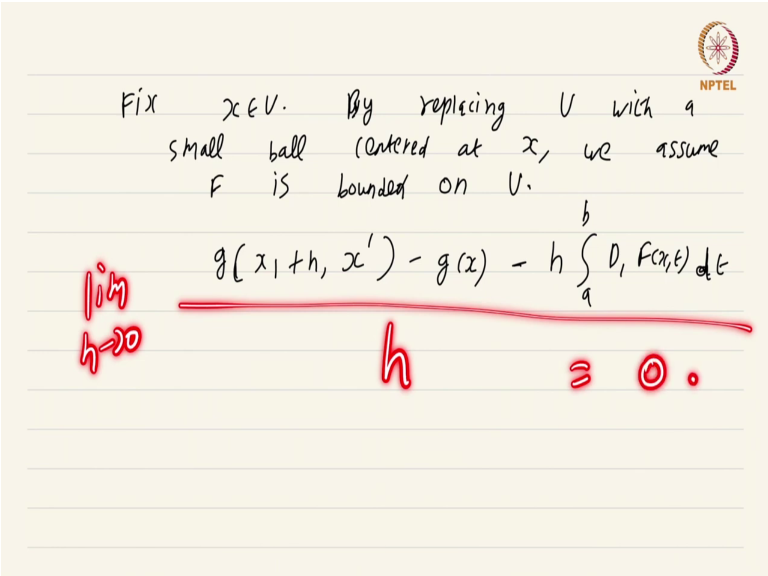

Fix x in U fix x in U ok now by replacing U with a small ball with a small ball centered at x centered at x, we assume F is bounded on U. Differentiating a function is a local thing the derivative of a function the existence and the value of the derivative depends only on the value of the function near that point. So, all we are going to do is we replace U by a small ball centered at x and we assume that F is bounded on U ok.

Now let us just compute and show this result for the first derivative. All the other derivatives following the exact same manner what we are interested in is g of x 1 plus h, the rest of the variables I am denoting by x prime x 2 to x n is denoted by x prime minus g x minus g x minus h times integral a to b D1 of F of x t d t ok.

So, I have just considered essentially the linear approximation I am just subtracting the linear approximation of g, considered now as a function of just the first slot all the other slots are fixed I am writing down the difference. Now for this function to be differentiable and the derivative to be this quantity integral a to b D1 of F of x t d t, all we have to show is this whole thing divided by h limit h going to 0 is 0. If you could show that we are done ok.

So, that is the aim so we will just analyze this quantity in some detail and see what happens ok.

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Fix $x \in U$. By replacing U with a small ball centered at x, we assume F is bounded on U.

$$g(x_1+h, x') - g(x) - h\int_a^b D_1 F(x,t)\, dt$$

$$= \int_a^b \left\{ F(x_1+h, x', t) - F(x,t) - h D_1 F(x,t) \right\} dt$$

$F(x_1+uh, x', t) : [0,1] \to \mathbb{R}$.

$$\int_0^1 \left[D_1 F(x_1+uh, x', t)\, h \right] du = F(x_1+uh, x', t)\Big|_0^1$$

Now, what I am going to do is we know the definition of g it is in terms of an integral involving F. So, we just write this as integral a to b F of x 1 plus h I forgot an x prime. So, F of x 1 plus h comma x prime comma t now because there is a t slot and in fact you are integrating along this t slot minus F of x t ok. and what I am going to do is I am going to just bring this derivative the candidate derivative also inside F of x t the whole thing d t integral from a to b ok.

So, this is just straightforward properties of the integral, by the way this integral I did not speak about that this integral exists this integral exists because D 1 of F of x t is a continuous function F. In fact, on U cross U and we are just fixing this x and treating it as a function of t and its certainly going to be continuous when you restrict it and treat it just as a function of t. So, this what is the integrand is actually continuous. So, this integral exists that needs to be said actually.

Now for the next step I am going to apply a trick, first observe that if you have consider the function F of x 1 plus U h comma x prime comma t, this I am going to treat it as a function of U ok. I am going to treat it as a function of U and this function I am going to assume that h is chosen so small that this as a function of U is defined from 0, 1 to R ok.

So, we are already taking h quite small we take h so small that this function treated as a function of U alone and as no other variable is well defined as a function from 0, 1 to R ok. We choose h so small that this happens ok. Now if you differentiate this function if you differentiate this function remember it is a function of U it is a one variable function.

You will get by the chain rule you will get D1 of F of x 1 plus U h comma x prime comma t and because we applied the chain rule you will get into h ok, this will be the derivative of this function. Consequently if I integrate this from 0 to 1 D u if I integrate this from 0 to 1 d u the derivative and the integral will just vanish, I will just get F of x 1 plus comma x prime comma t with the limits 0 and 1.

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$$F(x_1+h, x', t) - F(x_1, x', t).$$

$$= \int_a^b \left[\left(\int_0^1 D_1 F(x_1+uh, x', t)\, h\, du \right) - h D_1 F(x,t) \right] dt.$$

$$= \int_a^b \left[\int_0^1 D_1 F(x_1+uh, x', t) - D_1 F(x,t) \right] h\, du\, dt$$

$$\left| g(x_1+h, x') - g(x) - h \int_a^b D_1 F(x,t)\, dt \right|$$

Which is just F of x 1 plus h comma x prime comma t minus F of x 1 plus 0, so that is just F of x 1 plus x sorry F of x 1 comma x prime comma t right. So, I mean putting all this together this big expression that we have that we have to deal with I can replace these first two terms by this integral ok.

So, putting all this together I will just write equal to, so this equal to is technically following on from here it is following on from here and on to the next page. This equal to integral a to b integral 0 to 1 d 1 of F of x 1 plus U h comma x prime comma t h d U and this integral closes minus h D 1 of F of x t and this whole thing this whole thing is integrated with respect to t ok.

So, this is the net upshot of the observations that I made in the previous end of the previous slide, this is nothing but integral a to b. Now I am going to bring that second term h D1 F x t also inside noting that there is no U dependency there I can write this as integral 0 to 1 D1 of

F x 1 plus U h comma x prime comma t minus minus D1 of F of x t D1 of F x t this whole thing h D U d t.

I can do this because there is no U dependence on the second function. So, I can may as well integrate with respect to U, note that I am integrating from 0 to 1. So, it does not really it is a same as just writing h D1 F x t integral 0 to 1 h D1 F x t du is same as just h D1 F x t ok.

Now, we have to estimate the terms. So, we are interested in estimating norm of g of F x 1 h sorry it is this is not actually a norm this is just a modulus, we are interested in estimating g of x 1 plus h comma x prime minus g of x minus h integral a to b D1 of F of x t d t ok.

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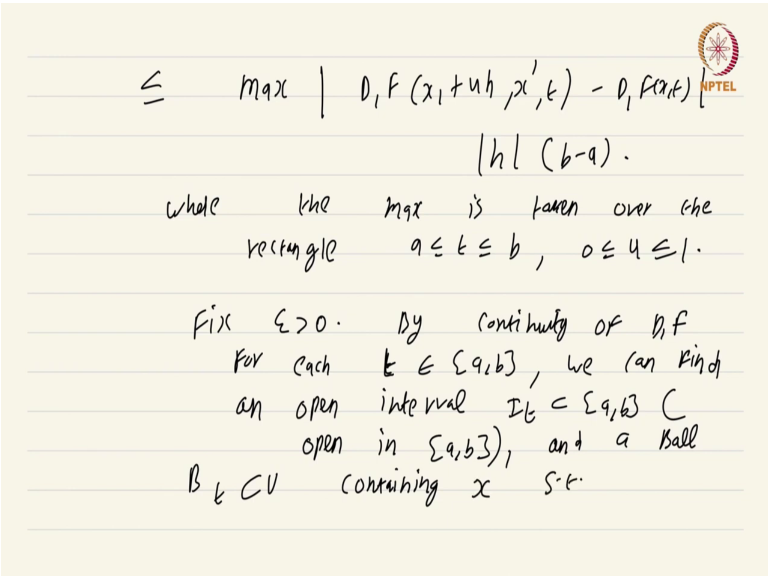

This is certainly going to be less than or equal to less than or equal to maximum value of the modulus of D1 of F of x 1 plus comma x prime comma t minus D1 of F of x t times the norm of h times the norm of h times the b minus times b minus a ok.

Where the maximum is taken over the rectangle a less than or equal to t is less than or equal to b and 0 is less than or equal to u is less than or equal to 1. This immediately follows from taking the modulus. So, we have to essentially we have to essentially take the modulus here to estimate this. So, we just move the modulus inside and just apply standard facts about integrals that we are already familiar with and from that we will get it.

So, just one error there is no norm on h there is just its just mod h this is just mod h right everything is just a real variable ok, this is a straightforward step I am going to leave the details to you. So, now fix epsilon greater than 0 ok. So, by continuity of D1 F for each t for each t in a b for each t in a b we can find; we can find an open interval an open interval I t which is a subset of a b. So, open in a b ok open in a b.

So that means this openness its not an open subset of R, but rather an open subset of 0, 1; please recall what this means by going through the relevant section in the chapter on metric spaces and a ball and a ball b t subset of u containing x such that the modulus of D1 of F of y s com minus D1 of F of x t is less than epsilon for all y in B t and s in I t.

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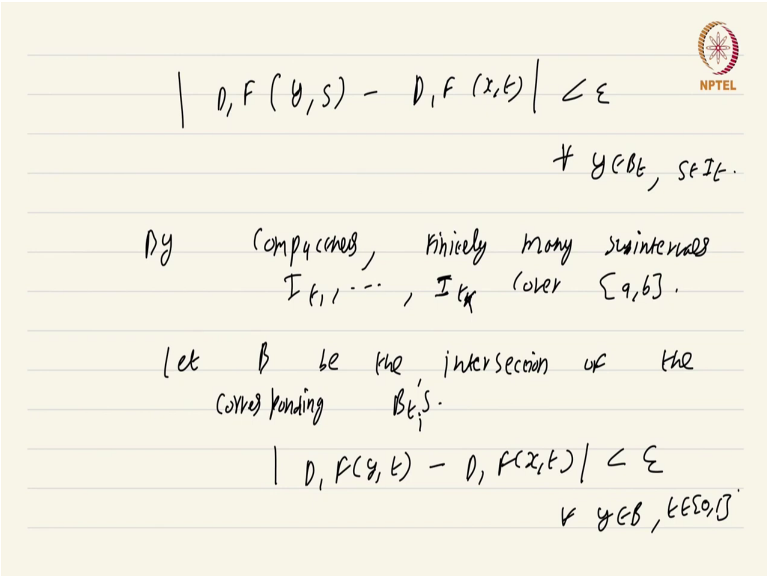

So, I have just applied the definition of continuity at the point x t, I am varying t for each t I will get an open interval I t subset of a b and ball b t that contains x such that this will be true this is just restating the definition of continuity for the function D1 of F ok.

Here is the key fact by compactness finitely many sub intervals finitely many sub intervals I t 1 to let us say I t k cover this close interval a b ok. Now, this is a somewhat tricky thing because the open intervals that I am considering are open in a b they are not open intervals of R, but recall from the chapter on metric spaces that compactness does not really depend on which ambient space you consider.

So, what I am saying is perfectly valid you can check it thoroughly please check that to make sure that you understand why this is correct. Let B be the corresponding or rather intersections be the intersection intersections intersection of the corresponding B t s ok or rather B t i s corresponding B t s that is B t 1 intersection B t 2 dot dot dot B t k.

So, we have covered a b by finitely many open intervals or rather yeah finitely many open intervals open in closed a b I reemphasize that. Now we are taking the intersection of the corresponding B t i s. So, once we take this intersection and apply this compactness argument. What we can conclude is that we get some sort of uniform continuity. What I mean by that is we can conclude that modulus of D1 of F of y t minus D1 of F of x t is less than epsilon for all y coming from the single ball B and t coming from 0, 1 ok.

So, by applying this compact as argument we get some sort of uniform continuity on the derivative D1. What this allows us to do this?

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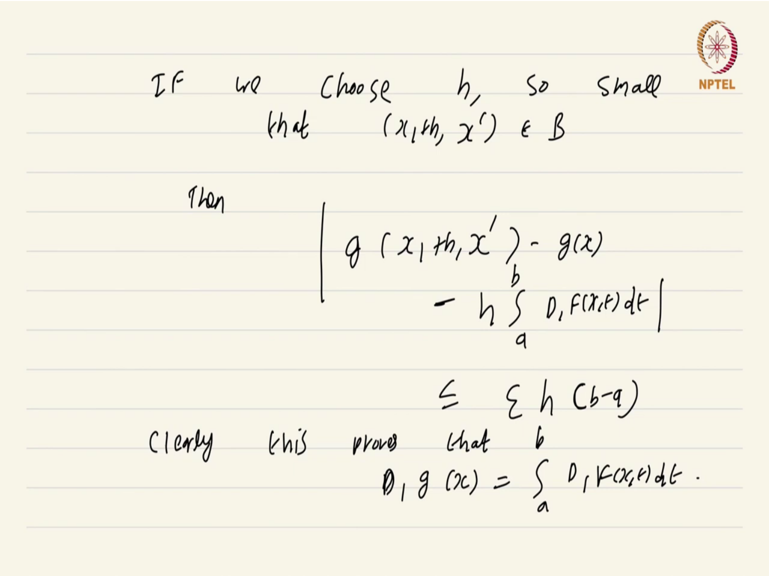

If we choose if we choose h so small that x 1 plus h comma x prime is always in B, then we get then we get the norm of g of x 1 plus h comma x prime minus g of x minus h integral a to b D 1 of F of x t d t. This whole thing it is not norm it is just modulus is less than or equal to epsilon times h times b minus a ok.

So, this just follows this just follows from the estimate we just got above that D1 of F of y t minus D1 of F of x t the absolute value of that is less than epsilon ok. And clearly this proves that D1 of sorry D1 of g correct D1 of g x is equal to integral a to b D1 of F of x t d t as claimed.

Well the other partial derivatives can be done in the exact same way. So, this was a somewhat tricky proof it is not hard its somewhat tricky because you have to apply this somewhat different type of compactness argument so, but essentially you get some sort of uniform continuity that allows us to do this. You can get a better argument for this if you invoke what is known as the dominated convergence theorem, that will allow you to prove this in a much easier way instead of this somewhat tricky argument.

This is a course on Real Analysis and you have just watched the video on Differentiating under the Integral sign.