Real Analysis II
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Lecture - 13.2
The Mean-Value Theorem

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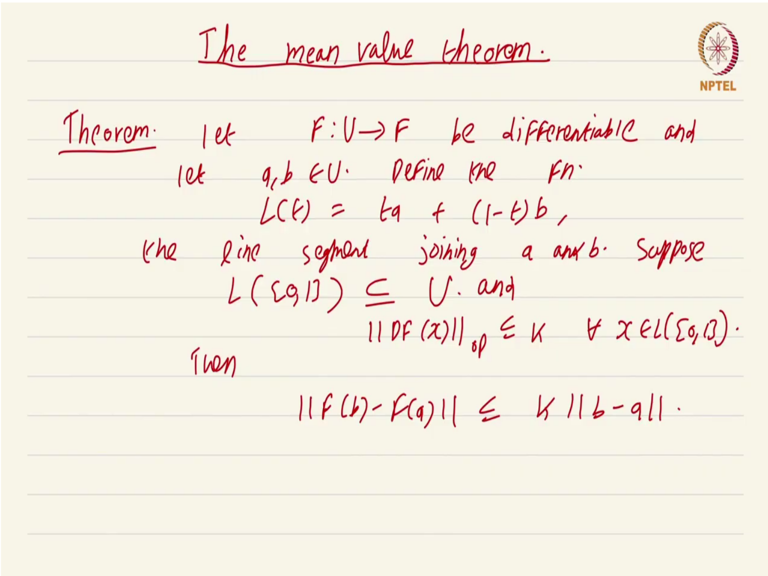

In this video we are going to now discuss the Mean Value Theorem in higher dimensions. As we have seen earlier the straightforward generalization of the mean value theorem is no longer true for vector valued functions of a scalar variable. The version I am going to present is actually an inequality and it works for differentiable mappings from any open set in R n to R m.

So, between arbitrary Euclidean spaces so the theorem is as follows theorem, let F from U to F be differentiable and let a comma b be points of U. Define the function; define the function

L of t to be nothing but t a plus 1 minus t b. This is nothing but the line segment joining a and b.

Suppose this a entire line segment L of 0, 1 is fully contained in U ok and furthermore assume that the operator norm of the derivative; the operator norm of the derivative is less than or equal to some constant K less than or equal to k for all x on L of 0, 1 ok. On this line segment suppose the operator norm there is a uniform bound capital K, then norm F of b minus F of a is less than or equal to K times norm b minus a.

So, there is now a inequality the usual mean value theorem said that there will be a point such that F of b minus F of a is equal to d F of x times b minus a. That is no longer true in general we have already seen a counter example, but this inequality will be more than sufficient in most applications.

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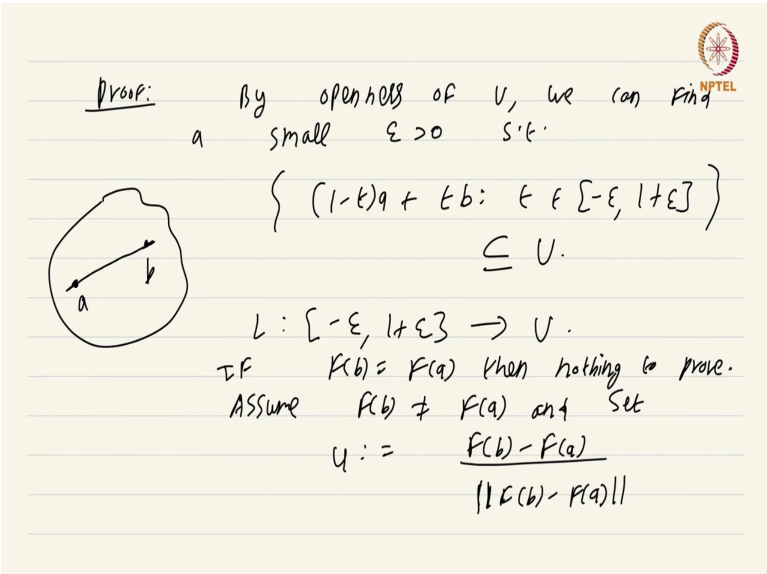

Let us see a proof. So, by openness of U we can actually find we can find a small epsilon greater than 0 such that this entire set 1 minus t a plus t b. Such that t comes from closed minus epsilon to 1 plus epsilon, this whole thing is a subset of U. That is we have assumed that the line segment, so if this is U and this is a and this is b.

So, this is a and this is b we have assumed that the line segment joining a and b is contained fully in U. what I am saying is you can extend this line segment a bit further on both sides. That is essentially what I am asserting and this will follow from the openness of U in a straightforward way. So, we can view this function L from minus epsilon 1 plus epsilon to U there is still no issue it will still map into U it will not take it outside U ok.

So, if F of b equal to F of a we are done nothing to prove then nothing to prove, so that takes care of the trivial case ok. So, assume F of b is not equal to F of a and set the vector U; set the vector U to be F of b minus F of a by norm F of b minus F of a that is the unit vector. The unit vector in the direction F of b minus F of a ok.

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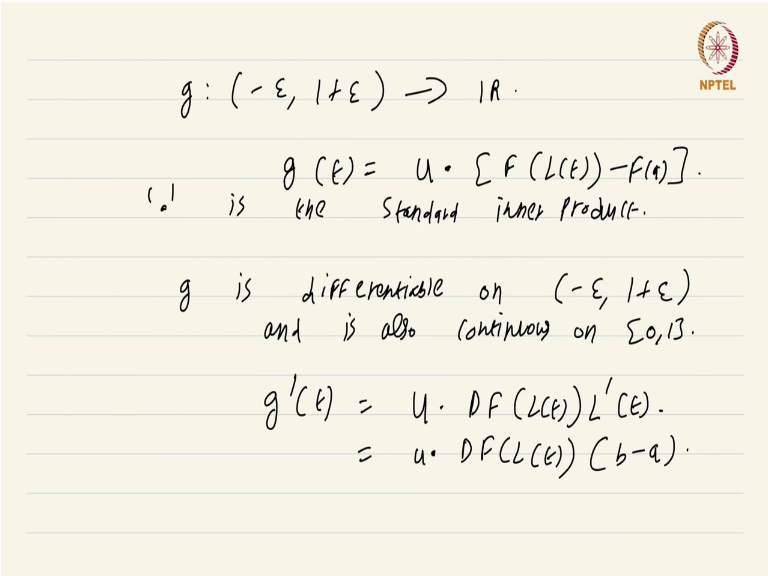

Now, we are going to define a new function g we are going to define a new function g from minus epsilon 1 plus epsilon to this is just going to map to the real numbers it is going to map to the real numbers; this is as follows g of t is just this vector u dot product the standard dot product F of L of t; F of L of t minus F of a ok.

So, here of course, dot is the standard inner product; the standard inner product. So, we have manufactured a function that maps this interval minus epsilon 1 plus epsilon to the real numbers ok. Now it is elementary that the function g is differentiable its differentiable this follows from the various properties of the derivative map that we have already established.

This proves that g I mean from those properties you can conclude that g is differentiable on minus epsilon 1 plus epsilon this is differentiable as a usual classical one variable function and is also continuous. Because it is differentiable is also continuous on close 0, 1 ok excellent.

And from the properties that we have seen it is easy to see that the derivative g prime of t is going to be u dot product DF taken at the point L of t and this DF taken at the point L of t acts on the vector L prime of t ok. This is going to be the derivative this is just an application of the product rule that we have seen and noting that u is just a constant its just a constant vector ok. So, this is a just equal to u of d a u dot product sorry u dot product D F L of t.

And remember L of t was 1 minus t a plus t b and the derivative of that is as a vector function is nothing but b minus a is nothing, but the constant vector b minus a. That is to be expected because L of t is actually just a line segment it parameterizes a line segment the derivative is b minus a ok.

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$$
\begin{aligned}
|g'(t)| &= |u \cdot DF(L(t))(b-a)| \\
&\leq \|u\| \cdot \|DF(L(t))(b-a)\| \\
&\leq \|DF(L(t))\|_{op} \cdot \|b-a\|. \\
&\leq K\|b-a\|.
\end{aligned}
$$

Apply MVT to g.

$$|g(1)-g(0)| = \|F(b)-F(a)\|$$

$$\|$$

$t_0 \in (0,1)$. $\quad |g'(t_0)| \times 1$

So, what have we got from all of this well we have got when by taking modulus on both sides or the absolute value we have got mod g prime of t is equal to the absolute value of u dot D F at L of t acting on b minus a acting on b minus a modulus ok. Now this by Cauchy Schwarz inequality is just less than or equal to norm u times the here this dot is just times ok it has nothing to do with the dot product DF L of t acting on b minus a ok.

And now by using the basic properties of the operator norm this is less than or equal to norm u, which I can get rid of because norm u is a unit vector remember that. So, this is less than or equal to D F L of t; L of t operator norm operator norm times norm b minus a ok. So, this is one of the properties of the I mean we have used the properties of the operator norm ok.

So, we get that mod g prime of t; mod g prime of t is less than or equal to the operator norm of DF at L of t times norm b minus a and now we can use this fact that the operator norm on L of t is bounded by this constant K to write that as K times norm b minus a. So, apply the classical mean value theorem apply the classical mean value theorem to g which we can because g certainly satisfies all the hypothesis. The end points we are choosing are 1 and 0.

So, mod g of 1 minus g of 0 recall that g was defined g was defined by this expression u dot F of L of t minus F of a and u was F of b minus F of a by norm F of b minus F of a. So, what happens is when you plug in t equal to 0 in this definition you get F of a minus F of a. Simply because L is supposed to be the line segment that joins a and b. So, you get a grand 0 and F of 1 is supposed to be F of L of 1 minus F of a which is just F of b minus F of a.

So, you got you have got u dot F of b minus F of a and u was just F of b minus F of a by norm F of b minus F of a. So, the net upshot is g of t minus g of 0 will be nothing but norm F of b minus F of a ok. So, what does the mean value theorem say well this is going to be modulus of g prime of t naught times 1 minus 0 which is just 1 where t naught lies somewhere in between 0 and 1.

So, there is some point in the open interval 0, 1 such that this happens. So, the net upshot of all this is the fact that combining I mean of course, combining these 2 combining these 2.

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$$\| F(b) - F(a) \| \leq K \, \|b-a\|.$$

We get norm of F of b minus F of a is less than or equal to K times norm b minus a as asserted ok. So, this concludes the proof of the mean value theorem. So, in the exercises you will show formulate a version of the mean value theorem that does not involve an inequality at least for real valued function.

Its not that hard you just look through the idea here you can also formulate a general version of the mean value theorem that does not involve in inequality by observing what we have done here. But that is not very useful this inequality final inequality that we get is more than sufficient for applications.

This is a course on real analysis and you have just watched the video on the mean value theorem.