Real Analysis II

Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 13.1
Properties of the Derivative Map

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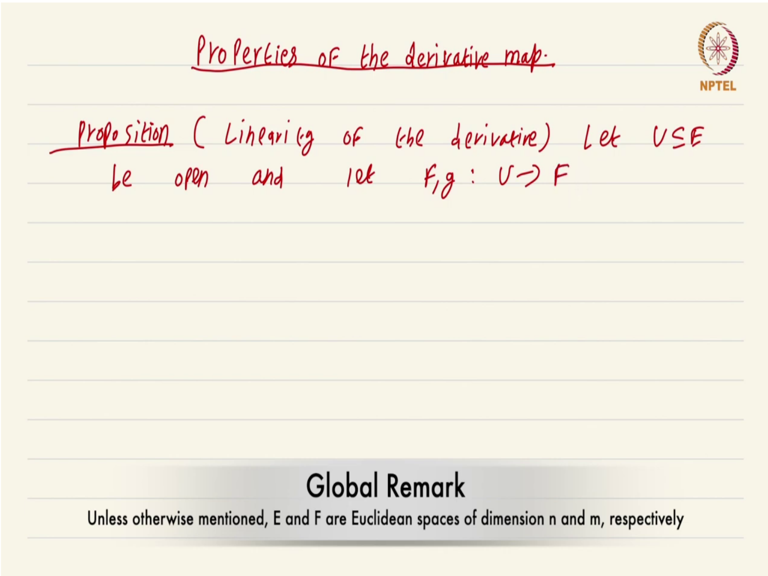

Let us discuss some basic Properties of the Derivative Map. The proofs of these are rather easy, they are modelled on similar proofs, we have been seeing from high school. First let us start with the proposition, this just says that, the derivative is linear in a different sense, linearity of the derivative.

We already know that the derivative is a linear map, but this is sort of saying that the taking the derivative and treating it as an operator that itself is linear. So, the setup is as follows, let

U subset of E be open and let f comma g from U to R m or rather F in our notation to F be differentiable at x in u.

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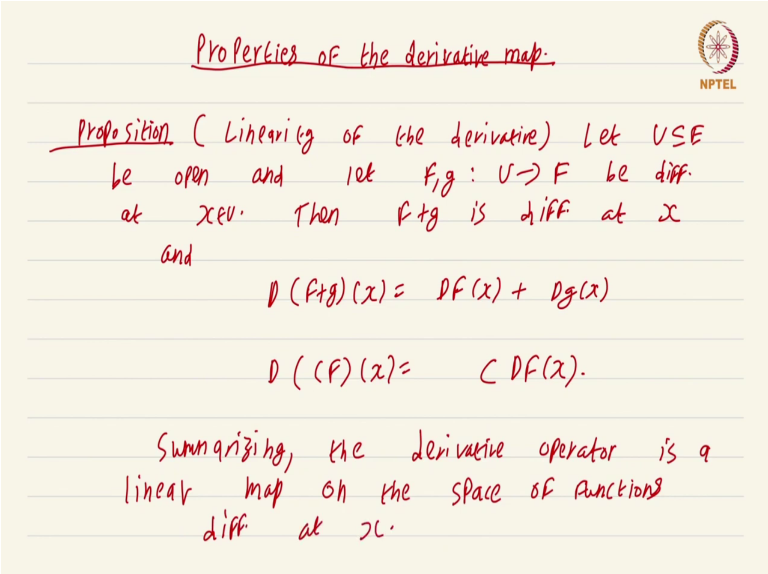

Then F plus g is differentiable at u, differentiable at x sorry at x and D of F plus g at x is just D F x plus D g x. And in the entirely similar way, D of C F at x is just C times D F x. So, treating the operator D has taking a function and outputting a linear map, that itself is linear on the space of differentiable functions.

So, summarizing, the derivative operator, the derivative operator is a linear map is a linear map on the space of; on the space of functions differentiable at x, ok. So, the statement is really long, the proof is going to be rather easy.

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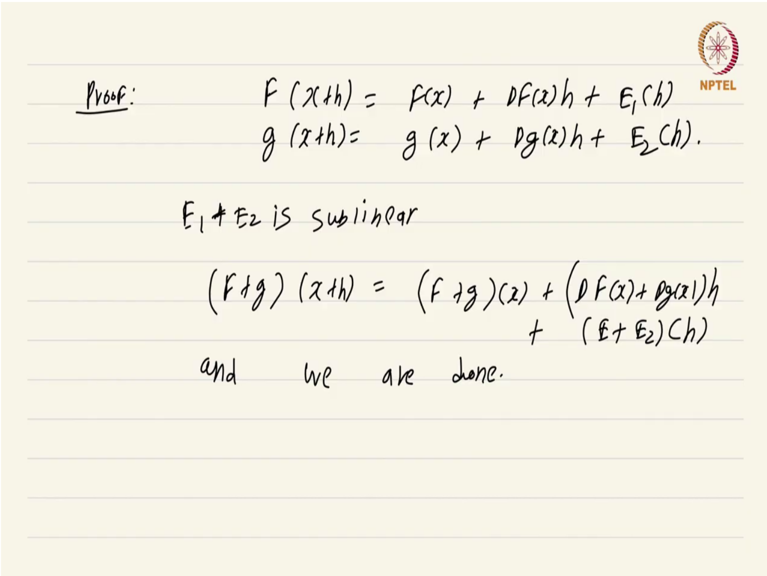

So, this is one of those scenarios, where all the work has been done over the years. So, what is the meaning of F and g are differentiable? We just write down F of x plus h is going to be equal to F of x plus D F x h plus an error term which we call E 1 of h and in an analogous way g of x plus h is g of x plus D g of x h plus E 2 of h, ok. Now, E 1 plus E 2 is obviously sublinear, the sum of two sublinear functions is going to be sublinear, E 1 plus E 2 sub linear.

So, immediately we get F plus g of x plus h is equal to F plus g x plus D F x plus D g x, the whole thing acting on h plus E 1 plus E 2 of h and we are done and we are done. This just the whole thing can be summarized by saying that the sum of two sub linear functions is sub linear. And the same thing is going to happen for the scalar multiplication case, I am going to leave it to you, it is an easy check, ok.

Now, the next thing is we have dealt with sum and scalar multiplication; the next question to be asked is, what about product? Well, the issue is there is no natural way to multiply two vectors in higher dimensions; you can do it in special cases there is a thing called the vector product, which you would have studied in school.

But I want to generalize this notion of product in the following way by defining what is called a bilinear map.

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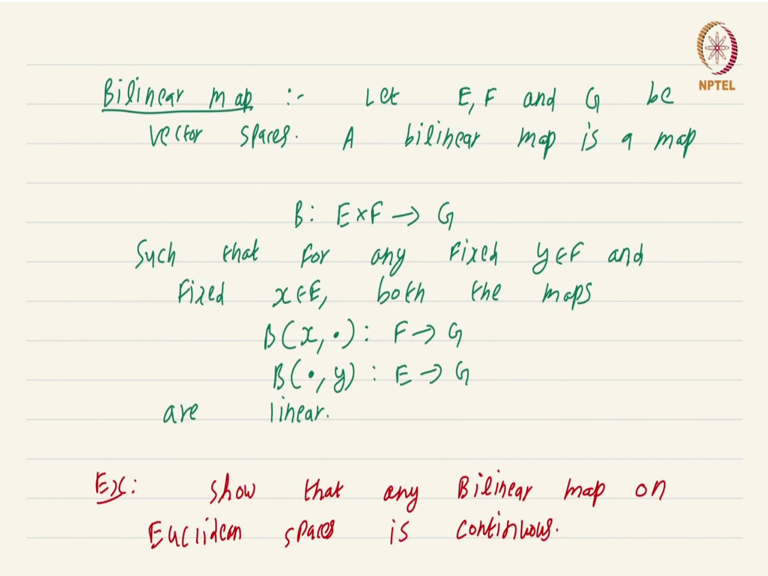

I am sure you are familiar with bilinear forms from your course on linear algebra, you might have studied bilinear maps also and the definition is not that hard. So, the setting is as follows. Let E comma F and G be vector spaces, be vector spaces. A bilinear map is a map is a map B from E times F, the Cartesian product of two vector spaces is in an obvious way a vector space; it is a map from the product E cross F to G, such that for any fixed y in F and fixed x in E, both the maps both the maps.

So, you fix the first slot to be x and treat the second slot as a variable, so you will get a map from F to G. And you treat the second slot, I mean you treat the first slot as a variable and the second slot is fixed; so you get a map from E to G, both these maps should be linear, such that both these maps are linear.

So, bilinear mapping is just a map that is linear when both variables are fixed. So, an easy exercise for you is the following; show that any bilinear mapping bilinear map on Euclidean spaces. So, E, F and G are Euclidean spaces on is continuous, bilinear maps are automatically continuous, ok.

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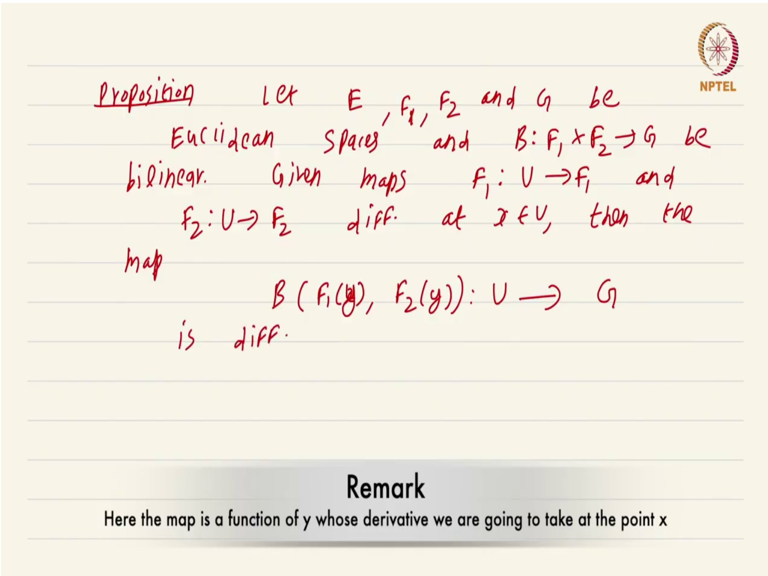

So, now with the notion of bilinear map at hand, we can state and prove a very simple product rule; it is quite general, but at the same time the proof is not hard. So, the setup is let E, F 1, F 2 and G be Euclidean spaces and B from F 1 times F 2 to G be bilinear, let this be bilinear, ok.

Now, given maps F 1 from U to F 1 and F 2 from U to F 2 differentiable at x in U, x in U; then the map; then the map B of F 1 x, not F 1 x, I will use F 1 y comma F 2 y, this is a map that starts in U and you land up in G, this map is differentiable at x. So, essentially we are taking a generalized product of F 1 and F 2 via this bilinear map B.

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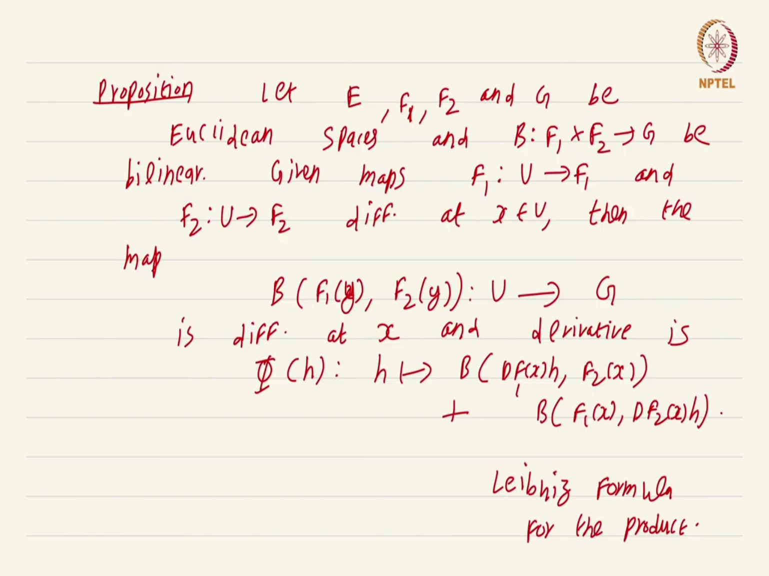

Is differentiable at x and the derivative surprisingly is given by phi of h and the derivative, derivative is phi of h takes h to B D F x h comma sorry F 1 x comma F 2 x plus B F 1 x comma D F 2 x h, ok.

So, this might look a bit surprising, but if you think about it, this thing whole thing is nothing but the standard Leibniz formula for the product; think about this for a while, why this is the standard Leibniz formula for the product. So, essentially the same product type rule holds even in this quite and abstract and general setting of bilinear maps and all that, ok.

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Proof: The claimed derivative map is indeed linear. Let E_1 and E_2 be the sublinear fns. coming from the defn. of diff. of f_1 and f_2, respectively.

$$B(f_1(x+h), f_2(x+h)) - B(f_1(x), f_2(x))$$
$$= B(f_1(x+h), f_2(x+h)) - B(f_1(x), f_2(x+h)) + B(f_1(x), f_2(x+h)) - B(f_1(x), f_2(x)).$$
$$= B(f_1(x+h) - f_1(x), f_2(x+h)) + B(f_1(x), f_2(x+h) - f_2(x)).$$

Now, let us prove this and the proof again is not that hard; the statement is quite complicated, but the proof is not hard. The claimed derivative map claimed derivative map is indeed linear; I want you to check this that is quite easy, you just have to use the fact that the place where h occurs there is a linear map in front of it and B of course is bilinear, so everything will go through, ok.

So, is the claim derivative map is indeed linear. So, let E 1 and E 2be the sublinear functions coming from coming from the definition of differentiability of F 1 and F 2respectively. So, F 1 and F 2 are differentiable. So, you have I mean, I am getting a bit bored of writing the same thing over and over again. So, you will have F 1 x plus h is equal to F 1 x plus D F 1 x h plus E 1 h so on and so forth, ok.

Now, we will have to do a straightforward, but somewhat long computation; each step is going to be utterly simple and modeled on the high school proof for the product rule. So, what do we have to compute? We have to compute B of F of x plus h, sorry F 1 of x plus h comma F 2 of x plus h minus B of F 1 x comma F 2 x.

We will have to compute this. And as usual we add and subtract an appropriate term to make things go through. So, we just write B of F 1 x plus h comma F 2 x plus h subtract B of F 1 x comma F 2 x plus h; then add back this term, which is B of F 1 x comma F 2 x plus h and finally, we add the original minus B of F 1 x comma F 2 x.

So, this is the standard add and subtract the combination of the quantities that you require, ok. So, now, this just is equal to B of F 1 x plus h minus F 1 x comma F 2 x plus h. What has happened? Well, these two terms, these two terms have been combined together by bilinearity; in a similar way, we are going to combine these two terms by bilinearity. So, you will get plus B of F 1 x comma F 2 of x plus h minus F 2 of x. So, we have just applied bilinearity twice to combine certain terms.

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$$B(DF_1(x)h,\ F_2(x+h)) + B(F_1(x),\ DF_2(x)h).$$
$$+\ B(E_1(h), F_2(x+h)) + B(F_1(x),\ E_2(h)).$$
$$=\ B(DF_1(x)h,\ F_2(x)) + B(F_1(x),\ D_2F(x)h)$$
$$+\ B(E_1(h), F_2(x+h)) + B(F_1(x),\ E_2(h))$$
$$-\ B(DF_1(x)h,\ F_2(x+h) - F(x)).$$

Sublinear

So, we have the term of the required shape. So, let us apply the definition of the derivative to both F 1 and F 2 and we immediately see that we get B of D F 1 x h comma F 2 of x plus h plus B of F 1 x comma D F 2 of x h, ok.

And of course, I must add the error terms. So, you will get B of E 1 h comma F 2 of x plus h, of course I am tacitly applying the bilinearity plus B of F 1 x comma E 2 h. So, I have combined two steps, I have expanded out F 1 of x plus h minus F of x and F 2 of x plus h minus F of x and applied bilinearity.

Now, notice that the first term is almost exactly what we want, except there is an additional h here which we do not want, ok. Other than that the first term is perfect and the second and third term, I mean the third and fourth, the third and fourth term are also almost exactly what we want.

So, we are going to show that, the third and fourth term are sub linear in any case; but before that we have to somehow fix the first term which has an additional factor of h. So, this is nothing, but doing the same trick again and again, this is the same add and subtract trick. You can check that this is nothing, but B of D F 1 x h comma F 2 x plus h F 2 x, sorry F 2 x plus B of F 1 x comma D 2 F x h plus B of E 1 h comma F 2 x plus h plus B of F 1 x comma E 2 h minus B of D F 1 x h, F 2 of x plus h minus F of x.

So, this last term is the thing that does the job for us; these two combine together, these two terms combined together would just give this term, ok. So, we have done the required manipulations; we have got the first two terms to be exactly what we claim is the derivative, which I have asked you to check is linear. So, our job is to now show that these three terms are sublinear.

So, I am going to say this final term, this final term sub linearity is very easy. So, this is sublinear is very easy, it just follows from continuity and linearity; it will just follow immediately from continuity and linearity that this term is sublinear, ok. So, we have left to check that these two terms are in fact sublinear. Let us just focus on one of these terms and show that it is sublinear, the other one is exactly going to be similar, ok.

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$$\frac{B(F_1(x), E_2(h))}{||h||} = B\left(F_1(x), \frac{E_2(h)}{||h||}\right)$$

$$\lim_{h \to 0} B\left(F_1(x), \frac{E_2(h)}{||h||}\right) = B\left(F_1(x), \lim_{h \to 0} \frac{E_2(h)}{||h||}\right)$$

$$= B(F_1(x), 0) = 0.$$

we are done.

So, let us focus on this second term B of F 1 x B of F 1 x comma D E to h sorry comma E to H. Let us try to show that this is sub linear; so we have to divide by norm h and see what happens as h goes to 0. Well by sub linear, by linearity of the bilinear map, I can just write this as B of F 1 x comma E 2 h by norm h, ok. Now, when I take limit h going to 0, I can push the limits all the way inside, simply because B is a continuous mapping. So, because B is a continuous mapping, I can put the limits inside.

So, this is nothing, but limit, not limit B of F 1 x comma limit h going to 0 E 2 of h by norm h. And this is nothing but B of F 1 x comma 0, which is just 0 by bilinearity, ok. In a similar way you can show that, the other term also goes to 0. So, we are done, we have shown that whatever remains is the error term, is nothing but a sublinear function, therefore we are done, ok.

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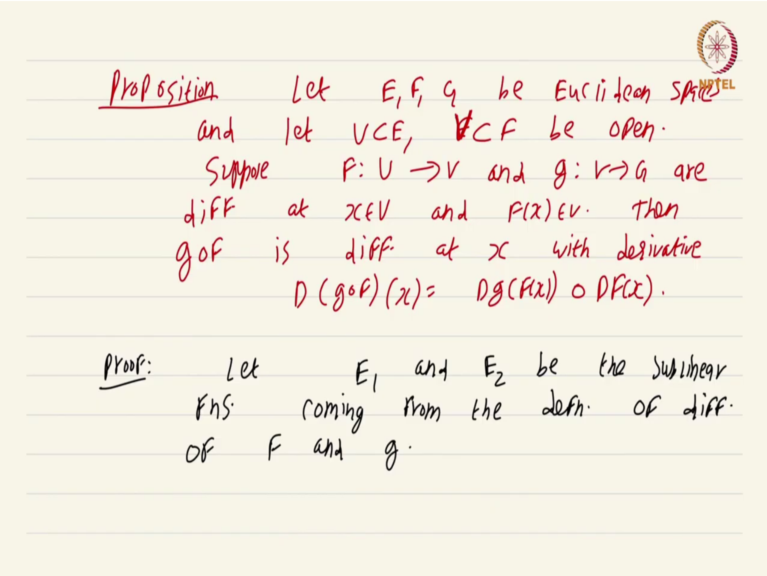

So, the product rule is also easily dealt with exactly like the way we did in high school. So, one final property of the derivative which is modeled on our high school studies is the chain rule. And the chain rule again the proof is going to be a bit tedious, but the basic idea is very simple. So, the setup is as follows.

Let E comma F comma G be Euclidean spaces, let them be Euclidean spaces and let U subset of E and F subset of sorry V subset of F be open, ok. Now, suppose F from U to V and g from V to G are differentiable at x in U and F of x in V; then g composed F is differentiable at x with derivative D g composed of D composed with F x is nothing, but D g of F x composed with D F x, this is the linear map which is going to be the derivative, ok.

So, the proof is going to be rather mechanical, but it is straightforward. So, what we are going to do is the following; let E 1 and E 2 be the sublinear functions the sublinear functions coming from the definition, coming from the definition of differentiability of F and g, ok.

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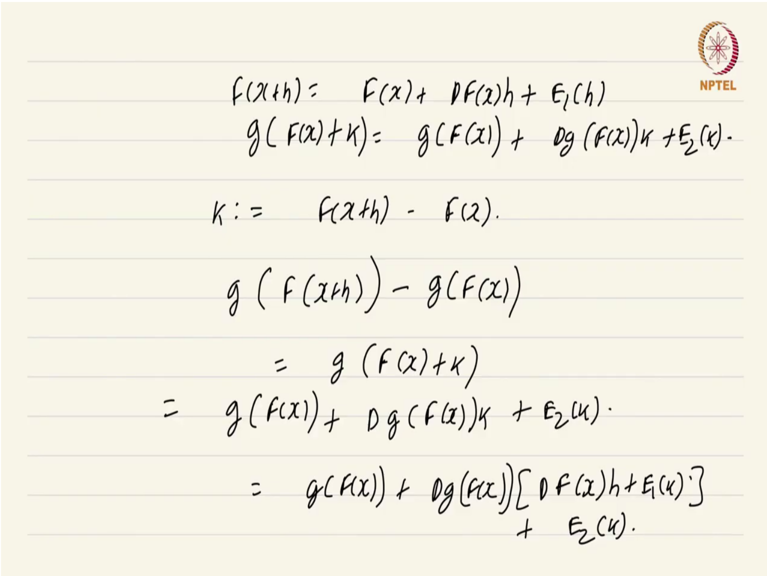

So, what this is essentially going to mean is that, F of x plus h is F of x plus D F x h plus E 1 h and g of F of x plus K is equal to g of F x plus D g at F x acting on K plus E 2 K, ok. Now, we are going to set K to be nothing, but F of x plus h minus F of x; as usual in the most proofs of chain rule, we have to do something like this and we have to compute what g of F of x plus h minus g of F of x is to get what we need, ok.

So, g of F of x plus h minus g of F x minus g of F x is nothing but g of F of x plus K; because K is nothing but F of x plus h minus F of F of x, F of x plus h is nothing but K plus F x, ok. So, this is a nothing but g of F of x plus k minus g of F of x ok, which is going to be or rather let me just write it as, this is equal to this is equal to; let me take it to the other side, this is equal to g of F of x plus D g at the point F of x acting on K plus E 2 of K, right.

So far nothing, but basic algebraic manipulations and this is nothing, but g of F of x plus D g F x and K is nothing, but F of x plus h minus F of x. So, this is nothing, but D F of x plus E 1 h; E 1 h ok; this is yeah D g of F of x acting on the vector D F x plus E 1 h plus E 2 of K plus E 2 of K, ok.

So, we have got something very similar to what we want; we have got this term g of F of x and we have got D g of F of x at the point F of x acting on D F x plus E 1 h plus E 2 of, yeah I missed a h, that is what that was what was throwing me off here, yeah this is actually h plus E 1 of K, yeah much better.

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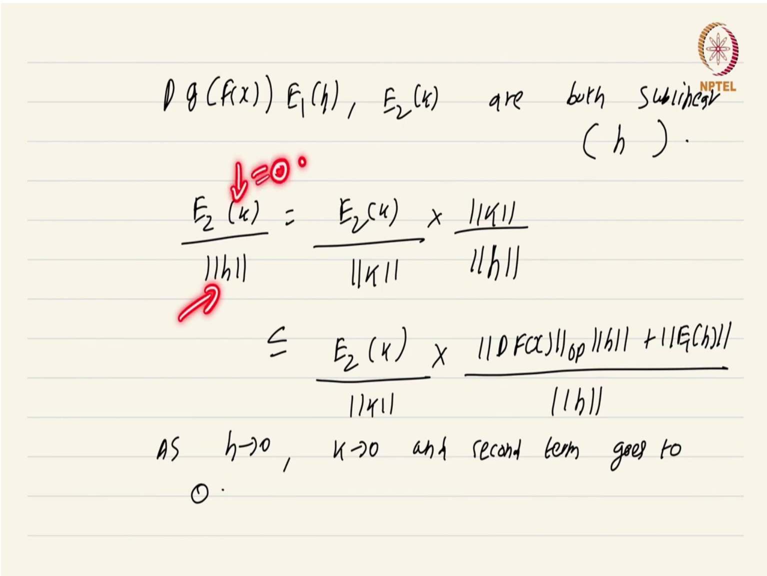

So, we have essentially got, what we need what we have to show is; we have to show that this term D g at the point F of x acting on E 1 h and E 2 of K are both sub linear are both sub linear.

And here the variable is h not k ok, with respect to h, ok. So, what we have to show is E 2 of k and E 2 D g F x at E 1 h are both sublinear. Let us take care of E 2 of k first. Let us look at E 2 of k by norm h, E 2 of k by norm h; this is nothing, but E 2 of k by norm k into norm k by norm h, ok.

And this is less than or equal to E 2 of k by norm k into norm D F x operator norm times norm h plus norm E 1 h divided by norm h. I have just expanded out what norm k is, which is F of x plus h minus F of x, which I am just expanding out by the definition of the derivative and then doing some basic properties of like triangle inequality and the basic properties of the operator norm, ok.

Now, as h goes to 0 as h goes to 0, this second term obviously goes to 0; the second term obviously goes to 0, ok. As h goes to 0, k goes to 0 as well and second term goes to 0; second term goes to 0 and because as h goes to 0, k goes to 0 by continuity of F at the point x, the first term also goes to 0, ok.

Now, there is one crucial thing that has to be remembered; we have made the assumption that k is not 0 as h approaches 0. But that is not really an issue, because what was the original expression that we were interested in; we were interested in this expression, E 2 of k by norm h. If for some value of h k happens to be equal to 0, E 2 of k is going to be anyway 0, ok.

So, as you approach h going to 0, we need to consider only those points where k is not 0 and estimate what happened, which is what we have done. So, this division and multiplication by norm k, which was crucial for the proof is actually justified; because at those points where k happens to be 0, we already know that this entire thing is 0, no more estimation needs to be done, ok.

Now, only one term remains that is this D g F x E 1 h; that this term is actually going to be sub linear is obvious and is left for you to do. So, the proof of the chain rule was a bit involved, but it is straightforward, you just have to be a bit careful; it is essentially one of those proofs, where you just write down what you have and just start working it out, you will eventually get the proof.

This is a course on Real Analysis and you have just watched the video on the properties of the derivative map.