Real Analysis II
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Lecture - 12.2
Examples of Differentiation

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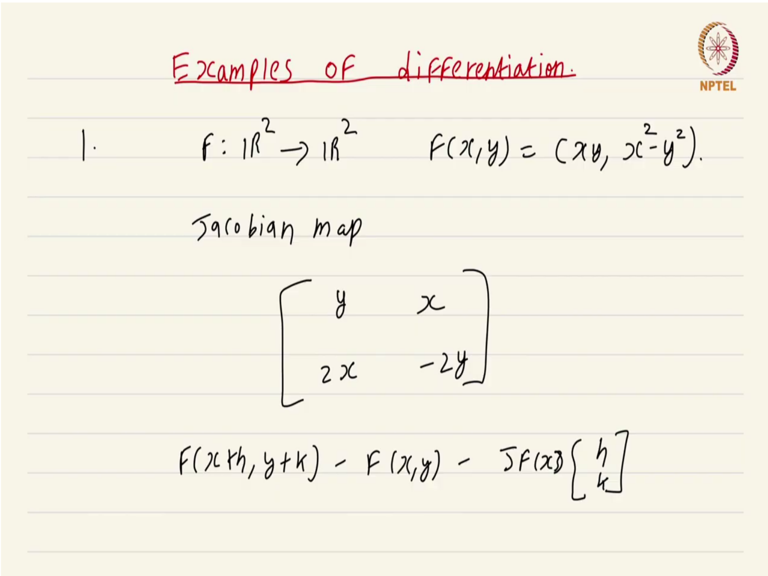

The notion of the derivative viewed as a linear map is very elegant, but because of the abstraction there is some difficulty in internalizing this concept. The best way to internalize the notion of the derivative as a linear map is to work out several examples. So, let us work out a few examples to get our hands dirty on computing the derivatives. Example number 1, let us start off with a very simple example.

So, we consider the function F from R 2 to R 2 given by F of x y is equal to x y comma x squared minus y squared. This is a rather simple example, in this scenario it is clear that each component function is in fact, differentiable, in fact, smooth all derivatives exist of each component function. Therefore, it is not a good idea to directly compute the derivative as a linear map rather its much easier to compute the Jacobian map

So, the Jacobian map, the Jacobian map is going to be clearly given by the derivative of x y with respect to x and y that will be the first row. So, that is going to be y x, then the derivative of x squared minus y squared with respect to x and y. So, that will be 2 x and minus 2 y ok. So, the Jacobian map is rather easy to compute and this gives us a readymade way of computing the action of the derivative on any vector. Let us now take it a bit further and actually try to compute; try to compute the error term.

So, the error term is going to be given by F of x plus h comma y plus k minus F of x y minus this Jacobian matrix F x acting on the column vector h k. So, this will give you concretely what the error term is going to be ok.

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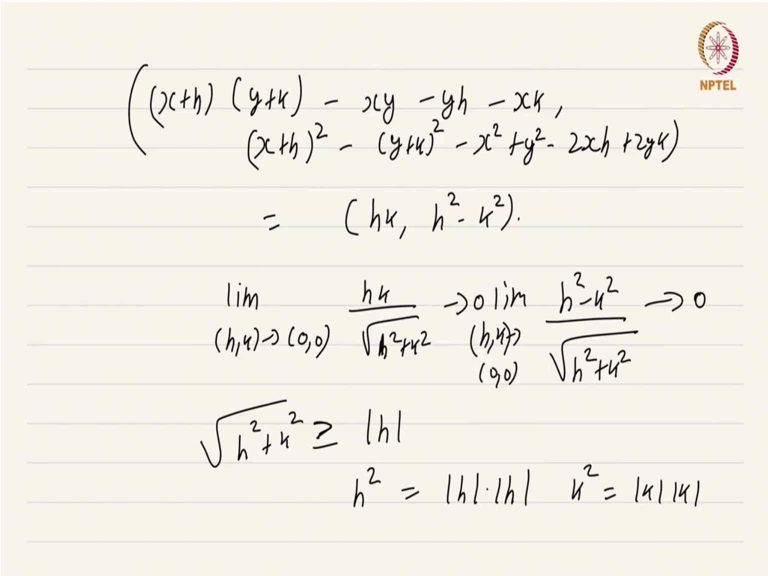

So, let us compute this out, this is going to be; this is going to be x plus h x plus h into y plus k minus x y minus y h minus x k. This is going to be the first component and the second component is going to be x plus h the whole squared minus y plus k the whole squared minus x squared plus y squared minus 2 x h plus 2 y k ok.

And lot of cancellations will happen and you will end up with h k comma h squared minus k square ok. Now let us verify that in fact, the error terms can go to 0. So, how do we do that? We have to show that limit h comma k; h comma k going to 0 comma 0 of h k by root of h squared plus k squared, this should converge to 0, and so much h squared minus k squared by square root of h squared plus k squared this must also go to 0 as limit h comma k goes to 0 comma 0.

This will show that the error term indeed behaves the way it is supposed to. We know that this is going to be true anyway, because the function, the components of the function are continuously differentiable, the partial derivatives exist and are continuous, but anyway let us verify it. Now to show that the first term is 0, its just we just have to use the fact that root of h square plus k squared is greater than or equal to is greater than or equal to mod h ok, and that will immediately show that the first term goes to 0.

And for the second term what we do is, we write h squared; we write h squared as mod h times mod h and mod h time mod h and k squared as mod k times mod k.

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$$\frac{|h^2-k^2|}{\sqrt{h^2+k^2}} \le |h|\cdot\frac{|h|}{\sqrt{h^2+k^2}} + |k|\cdot\frac{|k|}{\sqrt{h^2+k^2}}$$

$$\le |h|+|k| \to 0$$

as $(h,k) \to (0,0)$.

After writing h squared as mod h into mod h and k squared as mod k into mod k we can estimate mod h squared minus k squared divided by square root of h square plus k squared. This is going to be certainly less than or equal to mod h into mod h by under root h square plus k squared plus mod k into mod k by under root h square plus k squared. And this whole thing is less than or equal to mod h plus mod k which goes to 0 as h comma k goes to 0 0.

So, that takes care of the error term, the error term does indeed go to 0 and the Jacobian matrix does give the best linear approximation of the function near the point x. We knew that already by an abstract theorem, now we have seen with our own eyes that the error term does indeed go to 0 ok. Now this was a rather easy example, most functions that you are likely to encounter will not be this simple.

When you are differentiating abstractly and treating the derivative as a linear map your perspective sorts of sort of opens up and there are all sorts of new things that you can study from this perspective. So, we are going to take somewhat more detailed look at the derivative as a linear map by considering more sophisticated examples.

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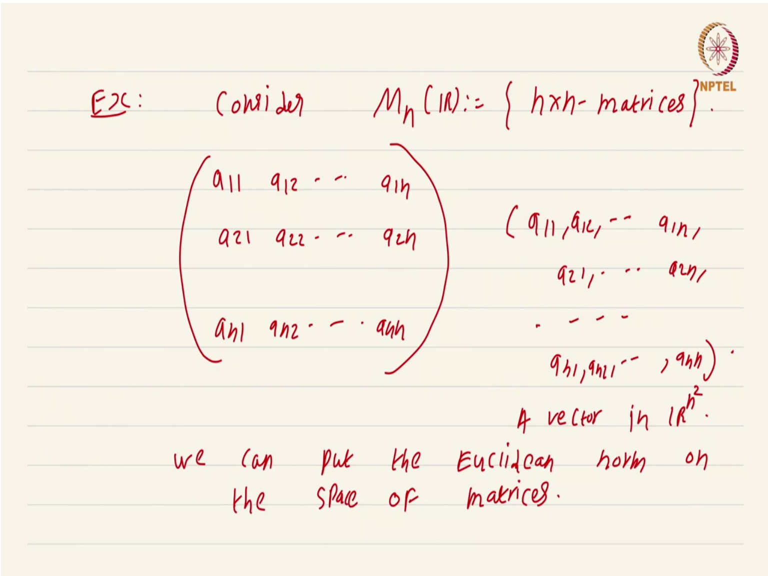

Consider, so I am going to give this as an exercise and you should solve this entire exercise in detail, because what follows will not make any sense without solving this exercise. So, consider M n R; this is n cross n matrices look at the collection of all n cross n square matrices with coefficients in the real numbers. Now a typical matrix, we can write it you will have a 11 a 12 dot dot dot a 1n a 21 a 22 dot dot dot a 2n then a n1 a n2 dot dot dot a nn ok.

Now what we can do is, we can just write this as a single row like this a 11 a 12 dot dot dot a 1n and then you start with a 21 a 22 dot dot dot a 2n comma dot dot dot a n1 a n2 dot dot dot a nn. So, this entire two-dimensional array of numbers you can represent it as a single linear array of numbers by just writing the rows one after the other to get a single row of numbers.

Obviously, you will get a vector of length. I will not use the word length because that is confusing, a vector in R n square that is a better way of saying unambiguous way of saying it. So, any n cross n square matrix over the real numbers can be viewed as a vector in R n square ok.

So, we can also put we can put the Euclidean norm now, the Euclidean norm on the space of matrices in this way on the space of matrices. So, this will make the space of matrices into a normed vector space, but there is a another viewpoint that we can take.

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Any $n \times n$ - matrix defines a L.T. on $\mathbb{R}^n$. We can put the operator norm also in this space.

$$\|A\|_{op} \leq \sum_j \sum_i |a_{i,j}|.$$

On any finite dimensional normed v.s., all norms are strongly equivalent.

Prove this directly for the operator norm and Euclidean norm.

Any matrix any n cross n matrix defines a linear transformation on R n right. So, I mean of course, I am going to view the matrix as being represented in the standard basis once after choosing the basis to be the standard basis in both the domain and codomain this matrix will represent a linear transformation.

So, we can put; we can put the operator norm also, the operator norm also in this space ok. Now the operator norm is much harder to compute in general, its not easy. So, one of the exercises is to show that you have a simple relationship, if you start with the matrix A the operator norm of this matrix is less than or equal to summation over j summation over i mod a i j; that is you are just summing up all the entries of the matrix that I mean in the representation.

I mean you are summing up over all the coefficients of the matrix A. You have also seen that on any finite dimensional space on any finite dimensional normed vector space, all norms are equivalent, all norms are strongly equivalent in fact, are strongly equivalent. So, technically the operator norm and the Euclidean norm will be strongly equivalent ok, but that is using an abstract fact prove this directly in the in this case prove this directly for the operator norm and Euclidean norm.

So, show that the operator norm and the Euclidean norm are equivalent on the space of n cross n matrices. So, why am I making these remarks, now we are going to be differentiating matrix functions that is functions that take a matrix and output another matrix for you, or functions that defined on a matrix that give you a number.

So, in some cases it will be convenient to use the operator norm. Note that the place where the norm enters the definition of the derivative is in the error term, more specifically when you are going to define sub linearity. It will not matter whether you use the operator norm or the Euclidean norm to define sublinearity, because both norms are equivalent.

So, in what follows I will use whatever is convenient, whichever norm is convenient for that particular problem I am going to use that particular norm ok.

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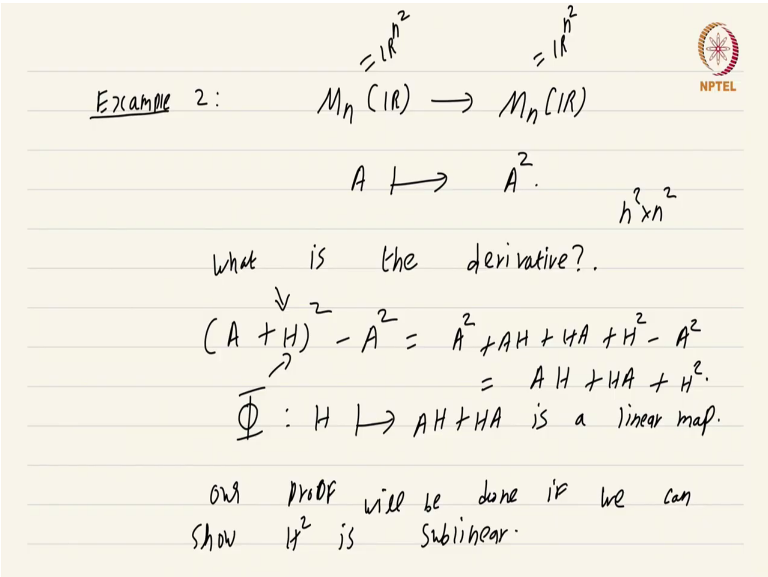

So, with all these preliminary set let us move on to example 2 and this is going to be a somewhat more involved example ok. So, example 2 is as follows; look at the space of matrices starting is MnR look at the map from MnR to MnR and the map is a matrix A goes to A square ok.

So, now of course, we have identifying this with R n squared and we are identifying this with R n squared, and the question is what is the derivative what is the derivative? A perfectly reasonable question to ask. Now here is a place where this idea of computing the Jacobian map or in other words computing the various partial derivatives is a futile effort unless you have an entire evening to waste. The reason is, look at the size of the resultant matrix not yeah the resultant Jacobian matrix, it will be an n squared cross n squared matrix that is huge.

I suggest that you do not do this, its not going to take that much time that saying that it takes an entire evening was for hyperbolic effect, but its going to be time consuming and its not going to give a lot of insight.

Its actually better to compute the derivative directly. But; however, first of all this map is differentiable and that can be seen by observing that once you write down the map in terms of R n square to R n squared, its very easy to see that this map when you write it in terms from R n square to R n squared each coefficient will be some polynomial in the entries right.

So, first check that in fact, this map is differentiable. So, it does make sense to talk about the derivative of this map. Anyway we will show as part of our computation for the derivative that this map is indeed differentiable, but its good to know that this map is differentiable directly, simply because when you write down in terms of this identification with R n squared each entry would be some polynomial in the original entries ok.

So, let us try to compute this directly. So, what must we do? We must compute A plus H the whole squared minus A squared this is what we are required to compute. So, this is A squared plus A H plus H A we have to be careful matrices do not commute plus H squared minus A squared which is A H plus H A plus H square ok.

Now observe that I have added I have incremented the matrix A by another matrix H. Now this is actually not an issue because when you add two matrices you add them coordinate wise anyway. So, when you add a matrix H whether you add it as matrices or you transform them into elements of R n squared and then add it really does not make a difference because addition is the same.

So, by this exact same explanation that I just gave, it is easy to check that the map phi that takes H to A H plus H A is a linear map; is a linear map. Its clear that its a linear map when you when we view the space as M n R when you when we view the space as M n R its very clear that this is going to be a linear map, but because what I said that it really does not matter whether you first do the addition as matrices and then transform to an element of R n squared or treat everything originally as an element of R n squared and then do the addition.

It really does not matter, because multiplying a matrix by a scalar or adding a two matrices is always coordinate why so? It really does not matter whether you identify with R n squared or not. So, we have got a linear map and we have got another term which looks peculiarly like an error term, because its H squared ok. So, all we have to do is to show that H squared is going to be sublinear ok.

So, our proof will be done our proof will be done will be done, if we can show H squared is sub linear ok. Let us do that, let us show that H squared is sub linear well for that its enough to show that norm H squared divided by norm H norm H goes to 0 as H goes to 0 ok.

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$$\|H^2\| / \|H\| \to 0 \quad \text{as} \quad H \to 0.$$

$$\leq \frac{\|H\|_{op}\|H\|_{op}}{\|H\|_{op}} \to 0.$$

$$\|AB\| \leq \|A\| \, \|B\|.$$

Now the this is clear for the operator norm. We have earlier shown that norm H squared is going to be less than or equal to norm H norm H and it will follow that norm H by norm H this goes to 0. But you can check that even in the Euclidean norm it follows that if you have two matrices norm AB is less than or equal to norm A norm B. This is true even in the Euclidean norm and its not very difficult to prove I suggest that you prove it. So, we actually do not need to go to the operator norm.

So, technically I must write up here. So, this inequality is not exactly correct, but under the operator norm this is true, it will be true under the Euclidean norm also. I leave these trivial checks to you ok.

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$$\|H^2\| / \|H\| \to 0 \quad \text{as} \quad H \to 0.$$

$$\frac{\|H\|_{op}\|H\|_{op}}{\|H\|_{op}} \to 0.$$

$\|AB\| \le \|A\| \|B\|$.

Example 3. $U \subseteq M_n(\mathbb{R})$ of invertible matrices

$GL_n(\mathbb{R})$.

$\det : M_n(\mathbb{R}) \to \mathbb{R}$. $\quad \det^{-1}(\mathbb{R} \setminus \{0\})$.

Now, let us take another example, this example is even more complicated and is again involves matrices ok. What we are going to do is, we are going to consider the subspace U of not subspace the subspace has a what do you say as a metric space U an open subset of M n R of invertible matrices.

So, in it the notation for this is often G L n R ok. The space of all invertible n cross n matrices. There are several ways to see that this is actually going to be a open set. One way is to consider the map determinant from M n R to R.

This determinate map is certainly a continuous map, because its a polynomial in the various entries of M n R and therefore, determinant is continuous and G L N R is just determinant inverse of R minus 0. R minus 0 is an open set not the inverse image of an open set is open, so this will show that G L n R is an open set ok. Let us see another proof of this fact that the space of all invertible n cross n matrices is actually an open set and here its very convenient to use the operator norm ok.

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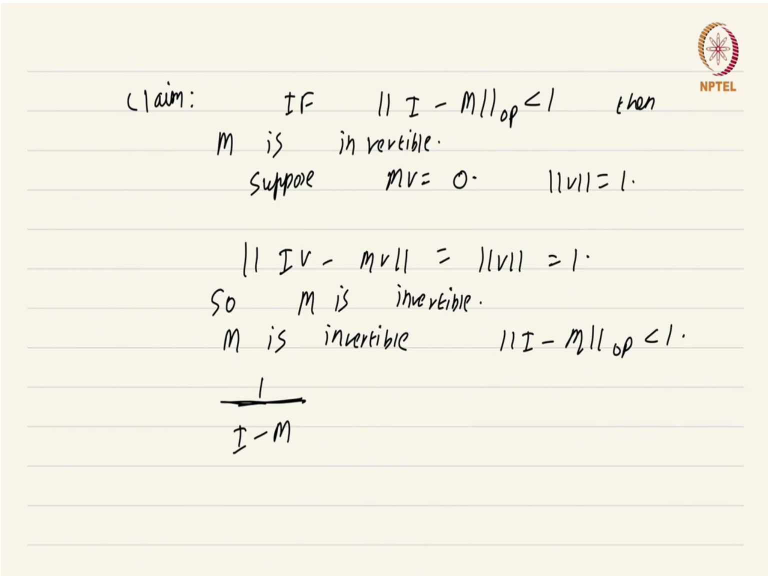

So, what we are going to do is the following; claim if identity minus M operator norm is less than 1 then M is invertible then M is invertible ok. Let us see why its true ok. Well suppose M v equal to 0.

That is the only way by which a self map of a finite dimensional vector space can fail to be invertible only if the kernel is non empty ok, then norm of Iv minus Mv is going to be equal to norm v because mv is 0. So, take of course, we can take norm v to be 1 we can take a unit vector that is there in the kernel, because the kernel is going to be a sub vector subspace.

So, I v minus M v equal to norm v which is equal to 1, but we just assumed that the operator norm of I minus M is less than 1 and we have got a unit vector such that norm of I v minus M v is equal to 1 which is not possible and this is a contradiction. So, M is invertible, so, M is invertible.

There is yet another way to see this which is important from a functional analysis perspective what we do is, we want to show that M is invertible we want to show that M is invertible and the given data is that I minus m operator norm is less than 1 ok. This is the data that we are given. Now the obvious thing to do would be to write down what 1 by I minus M is. Of course, 1 by does not make sense it is because we are in matrices.

But we are going to be a bit formal and claim that 1 by I minus M is actually nothing not 1 by I minus 1 what I am going to claim is I am going to claim that M.

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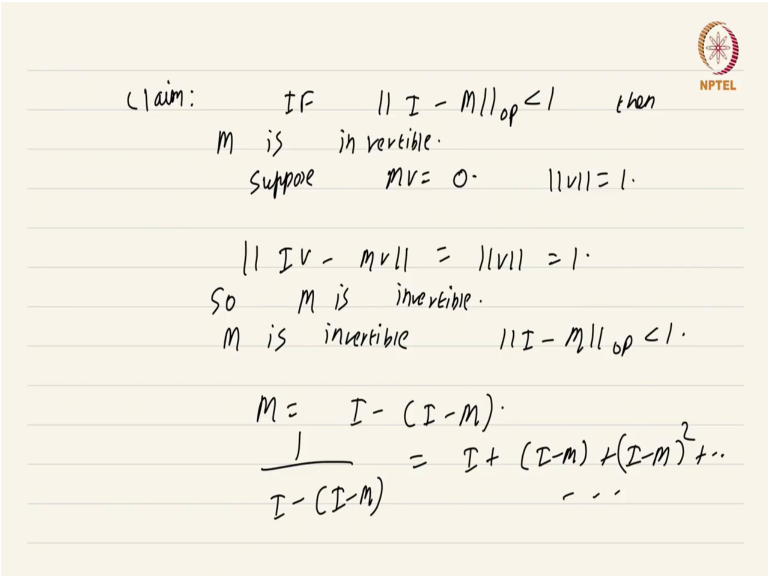

Which is actually I minus I minus M which is actually I minus I minus M I want to invert this that is I want to invert 1 by I minus I minus M. And I am just going to formally think that this is going to be I plus I minus M plus I minus M squared and so on right. Why do I think this is the case because this is just the formula for the sum of a geometric series, but what I have done in this entire stage, this expression is meaningless this is just; this is just formally ok

So, we are going to claim that this series that is there on the right-hand side; a series of matrices this is going to converge and we are going to claim that it converges to the limit ok. Now how why do we know that this converges.

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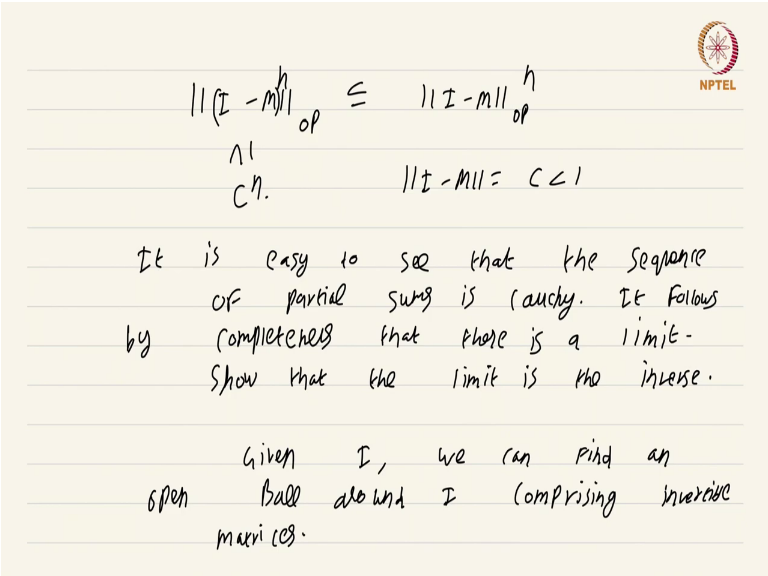

Well because norm I minus M operator norm power n or rather I minus M power n operator norm is norm I minus M operator norm power n. So, if norm I minus M is equal to C is less than 1 this is less than or equal to C power n ok.

Now, it is this shows this is easy to see now, it is easy to see, it is easy to see that the series is Cauchy or rather the sequence of partial sums, the sequence of partial sums is Cauchy. And as R n squared is actually complete it follows that there is a limit, it follows by completeness by completeness that there is a limit there is a limit ok.

So, show that show this show that the limit is the inverse limit is the inverse, and also as part of this show that this convergence happens in the Euclidean norm also. We know that because the operator norm and the Euclidean norm are equivalent, but just in case just make sure that you understand that it happens ok.

So, what is our conclusion? Well our conclusion is given, I mean given I we can find; we can find an open ball around I comprising invertible matrices; comprising invertible matrices ok.

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If $A \in GL_n(\mathbb{R})$, then we can find a small ball around A consisting only of invertible matrices.

$(A+H)^{-1} - A^{-1}$ $\qquad \|I - M\|_{op} < 1$

$= \left(A\left(I + A^{-1}H\right)\right)^{-1}$

Choose H so small that $\|A^{-1}H\|_{op} < 1$

$(I + A^{-1}H)$ is invertible.

Now, use this to show that if A is in G L n R if A is in G L n R we can find then we can find; we can find a small ball around A consisting only of invertible matrices ok. This is also left as an exercise for you ok. So, these were just some preliminaries about the inversion map [laughter]. Now is the real challenge, [laughter] we have to compute the derivative and the computation is somewhat tricky not too tricky, but somewhat tricky.

So, what is it that we have to do? We have to do e a of A plus H inverse minus A inverse. So, what was the strategy in the last problem well we just computed and ended up with the clearly linear term and some term which was quadratic which we guessed is the error term and showed that is the error term, we are going to try something similar here ok. Now, but there is a little bit of tricks involved in this. Now A is invertible, so what we are going to do is, we are going to write this A plus H inverse as A inverse into I plus A H and this whole thing is inverted ok.

Now, choose H so small that that norm. One moment I think I made a slight mistake here this is not, this is A into I plus A inverse H I completely reversed what I wanted to write this is A into I plus A inverse that is so small that A inverse H operator norm is A inverse H operator norm is less than 1 ok. Now this means I plus A inverse H I plus A inverse H is invertible ok, that is just by the discussion that we had that if I minus M operator norm is less than 1 then M is invertible.

Here I have that A inverse H operator norm is less than 1. So, I plus A inverse H will be invertible ok. And what is its inverse well its this infinite series which is going to be very complicated the inverse.

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$$(I + A^{-1}H)^{-1}$$

$$= (I - (-A^{-1}H))^{-1} =$$

$$= I + (-A^{-1}H) + (-A^{-1}H)^{2} + \cdots$$

$$(A+H)^{-1} - A^{-1} = A^{-1} - A^{-1} - A^{-1}HA^{-1} + ((-A^{-1}H)^{2} + (-A^{-1}H)^{3} + \cdots))A^{-1}$$

$$= \underbrace{-A^{-1}HA^{-1}}_{\text{linear}} + \underbrace{((-A^{-1}H)^{2} + (-A^{-1}H)^{3} + \cdots)A^{-1}}_{E(h).}$$

I plus A inverse H inverse is going to be nothing, but I minus A inverse H the whole inverse. Yeah I left out a minus sign its going to be I minus A inverse H the whole inverse which is going to be I minus A inverse H, then plus minus A inverse H squared and so on ok it will be better to actually write it for consistency plus here and minus inside, so that it looks consistent ok.

So, far so good a bit complicated, but nothing too hard. So, A plus H the whole inverse minus A inverse is going to be A inverse minus A inverse ok minus A inverse H A inverse plus plus minus A inverse H the whole squared plus minus A inverse H the whole cube plus dot dot dot into A inverse ok. I hope I have not made any silly mistakes I hopefully I am not ok. So, what is the net upshot of all this. Wait I have made a silly mistake this A inverse should be outside ok excellent.

So, this is going to be nothing, but minus A inverse H A inverse plus plus a bunch of complicated looking terms. So, minus A inverse H the whole squared plus minus A inverse H the whole cube plus dot dot dot A inverse ok. So, this part is going to be linear which I am going to leave it to you to check it straight forward. And this part which I am going to call E of H we have to show is sub linear ok.

So, check that this first map minus A inverse H A inverse is in fact, a linear mapping. All we have to now show is this complicated expression which we have called E of H is going to be sub linear ok. And of course, we are going to use the operator norm, because its much simpler that way.

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$$\|E(H)\|_{op} \le \|A^{-1}H\|_{op} \times \|-A^{-1}H + (-A^{-1}H)^2 + \cdots\|_{op} \times \|A^{-1}\|_{op}$$

$$\|-A^{-1}H + (-A^{-1}H)^2 + \cdots\|_{op} \le \|A^{-1}H\|_{op} + \|(A^{-1}H)\|_{op}^2 + \cdots$$

$$= \frac{\|A^{-1}H\|_{op}}{1 - \|A^{-1}H\|_{op}}.$$

So, norm of E of H the operator norm of this is going to be nothing, but less than or equal to A inverse H operator norm into. I am using some basic properties of the operator norm into norm of minus A inverse H plus minus A inverse H squared plus dot dot dot operator norm close into norm A inverse operator norm ok. Now you have this huge expression here you have this huge expression here.

Now, it is straightforward to check that this huge expression, this term alone, this expression is going to be less than or equal to less than or equal to. So, let me for clarity write the whole thing again norm of minus A inverse H plus minus A inverse H squared plus dot dot dot operator norm is less than or equal to norm A inverse H operator norm plus norm of A inverse H operator norm squared plus dot dot dot.

So, I have sort of applied a generalized triangle inequality. I am saying that if you have a convergent series and you have you can make a triangle inequality involving all the infinitely many terms ok. And now because this norm of A inverse H is a really small term we have taken it to be less than 1, we can sum up this series of real numbers using the geometric series expression. This is just norm of A inverse H operator norm divided by 1 minus norm A inverse H operator norm I am just applying the sum of a geometric series.

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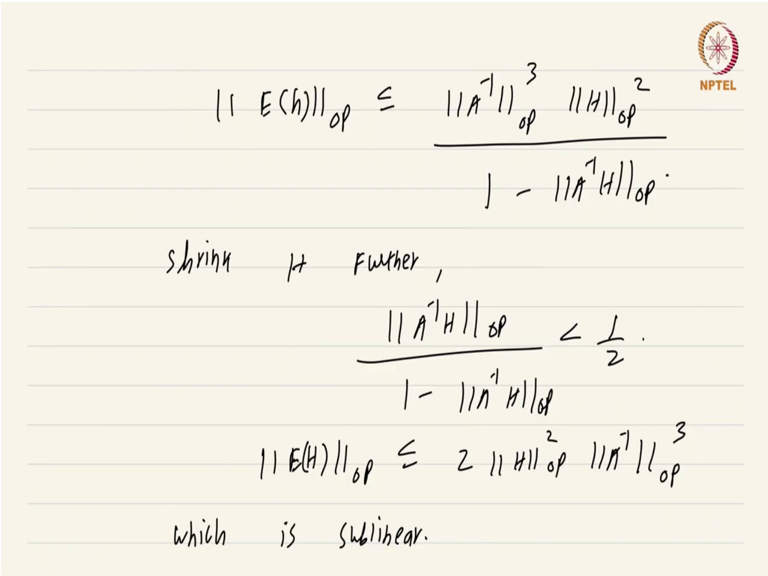

So, putting all this together we will get that the error terms operator norm is less than or equal to, I hope I get this correct the operator norm of A inverse cubed times norm H operator norm squared divided by 1 minus norm A inverse H operator norm ok. Now shrink H further that is make H closer to 0 to make sure that norm of A inverse H operator norm by 1 minus norm A inverse H operator norm is less than half. This can be certainly done

Once you do this you will get that the norm of E of H operator norm is less than or equal to 2 times norm H squared the operator norm my into A inverses operator norm cube ok which is sublinear clearly. Hopefully I have not made any errors, even if I have these errors will not really matter for the qualitative nature of the proof, ultimately we get that E of H is sub linear. So, this was a difficult example involving the computation of a derivative map ok.

So, let me summarize by saying that there are only three real possible ways of computing the derivative map. If the situation is very simple map between some R n and R m let us directly compute the Jacobian map. If its some complicated situation where computing the Jacobian map is not practical then one strategy is to try and guess the derivative map. This works for very simple examples, but it often does not work.

The third way is to directly start with the in increment, you want to do F of x plus h minus F of x. Write down the expression, collect together terms that look linear together and the rest of the terms call it the error term and somehow show that the terms that you have grouped together as the error term is sub linear.

So, this involves a bit of trial and error sometimes you might get the grouping wrong and so on, but it is essentially doable. These are the only three ways by which you can practically compute the derivative of a linear derivative of a map from R n to R m. This is a course on real analysis and you have just watched the video on examples of differentiation.