Real Analysis II
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Lecture - 1.4
Loads of Definitions!

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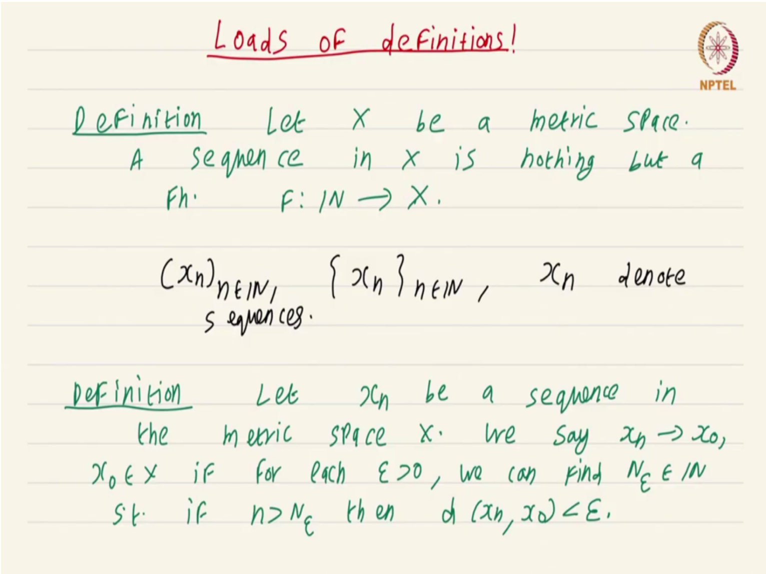

With the basic examples of metric spaces now well understood, let us proceed and give plenty of definitions. I urge you to connect each concept introduced in this module with the corresponding concept defined for the real numbers in the earlier chapter entitled a Taste of Topology. So, the first definition is going to be that of a sequence.

Definition: Let X be a metric space. A sequence in X is nothing but a function $f: \mathbb{N} \rightarrow X$. As usual, we shall use the notations $(x_n)_{n\in\mathbb{N}}$ or $\{x_n\}_{n\in\mathbb{N}}$ or just plain x_n to denote sequences.

Now, let us come to the most important definition of convergence. Again, it will be no surprise because we have already seen and spent considerable time digesting this definition.

Definition: Let x_n be a sequence in the metric space X. We say x_n converges to x_0, where $x_0 \in X$, if for each $\epsilon > 0$, we can find $\mathrm{N}_\epsilon \in \mathbb{N}$ such that

$$\text{if } n > N_\epsilon, \text{ then } d(x_n, x_0) < \epsilon.$$

This definition is saying that whenever this n is chosen suitably large, then the distance between x_n and x_0 can be made arbitrarily small.

The definition is exactly analogous to what we have already seen, digested, and assimilated in our bloodstream, the earlier concept of convergence that we saw for real numbers.

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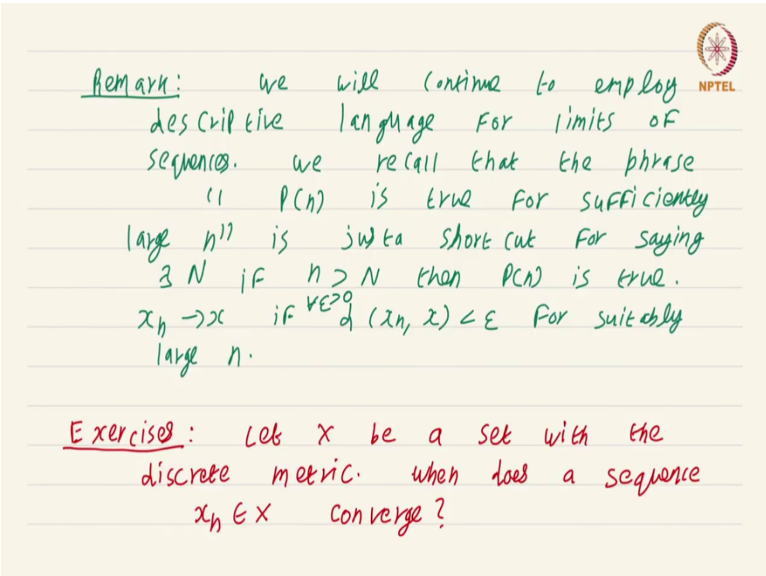

Now, let me just remark.

Remark: We will continue to employ descriptive language for limits of sequences.

What do I mean by continuing to employ descriptive language for limits? Well, I urge you to look through the corresponding section in the chapter on sequences and series that we saw before in real analysis I. But let me just recall one major phrase that comes up repeatedly that it is worth rewriting once more, even if it is going to be a repetition.

Remark: We recall that the phrase $P(n)$ is true for sufficiently large n is just a shortcut for saying $\exists\, N$ such that if $n > N$ then $P(n)$ is true. So, the tedious and complicated expression is shortened to saying $P(n)$ is true for sufficiently large n. So, the definition of convergence just reads x_n converges to x if if for all $\epsilon > 0$, $\mathrm{d}(x_n, x) < \epsilon$ for suitably large n.

So, I urge you to go back to the section on sequences and series in the chapter on topology on real numbers and revise all the basic concepts that we have introduced. It is going to be used without further description in what follows.

Now, what we are going to do is study some more definitions. Still, it might be worthwhile to pause and work out a couple of exercises to make sure that you understand what exactly is happening in this more abstract and general setting of metric space.

Again, to ground our knowledge, it might be a good idea to look back in the corresponding chapter on real numbers whenever a new definition is introduced. But of course, seeing that concept in real numbers will not enable you to understand and appreciate this more general theory fully. So, we should look through some examples that are far away from real numbers so that we have a good understanding of these definitions.

So, let me give you an exercise that will cement your understanding of sequences and convergence.

Exercise: Let X be a set with the discrete metric; that means X is a trivial metric space. When does the sequence $x_n \in X$ converge?

Think about a discrete metric space and consider a sequence x_n. In this discrete metric space, can you precisely describe when the sequence will actually converge? Is there any easy way to just look at the sequence and tell whether it converges or not? Solve this exercise.

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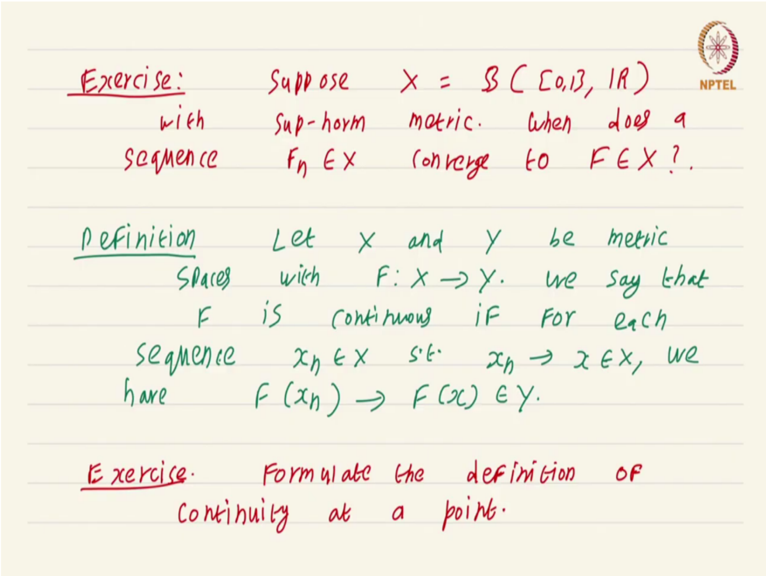

Next, another exercise,

Exercise: Suppose we consider the metric space $B([0,1], \mathbb{R})$. So, this is the metric space of bounded functions from [0, 1] to $\mathbb{R}$ with the sup norm metric. When does a sequence, when does a sequence $f_n \in X$ converge to $f \in X$?

So, please solve these two exercises. It will give you a grounding in the notion of convergence in this more general setting of metric spaces. Onwards definition and this is an important definition of continuity, but as we have digested this definition, it is not going to prove that much of a challenge.

Definition: Let X and Y be metric spaces with a function $f : X \to Y$. We say that f is continuous, if for each sequence $x_n \in X$ such that $x_n \to x \in X$, we have $f(x_n) \to f(x) \in Y$.

So, a function from a metric space X to a metric space Y is said to be continuous, if whatever sequence x_n you take that converges to some $x \in X$, we have that the image sequence $f(x_n)$ converges to the limit $f(x)$. If, you recall this was one of the characterizations of continuity that we have already seen for continuous functions from a subset of real numbers to real numbers.

We have already seen several characterizations of continuity, and this is one of them ok. In a later module, we will characterize continuity in terms of open sets as well. So, for the time being, you can solve one exercise;

Exercise: Formulate the definition of continuity at a point.

I will leave it to you to figure out what exactly this question is asking you to do. The moment you figure out what this question is asking you to do, actually solving the exercise will be over. So, I do not want to spoil the fun, so please think about this and do it. Some more definitions.

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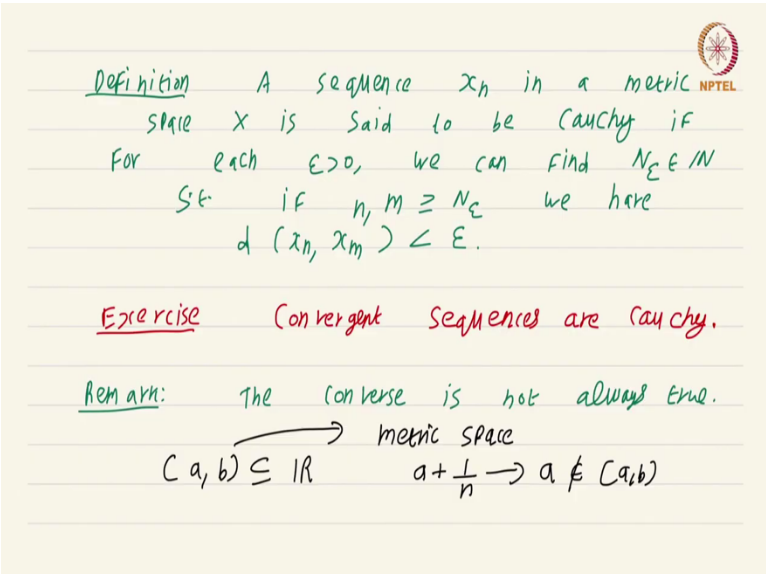

So, as I mentioned, this will be a long list of definitions, most of which you have already seen in the chapter on the topology of real numbers.

Definition: A sequence x_n in a metric space X, X is said to be Cauchy if, for each $\epsilon > 0$, we can find $N_\epsilon \in \mathbb{N}$, such that if $n, m \geq N_\epsilon$, we have $d(x_n, x_m) < \epsilon$.

So, this just says that the distance between terms of the sequence becomes arbitrarily small if n, m are sufficiently far. So, intuitively a Cauchy sequence is one in which the terms of the sequence start getting closer and closer to each other as we move towards the tail of the sequence. The same proof that you have seen will show that convergence sequences or Cauchy, which I am going to leave as an exercise;

Exercise: Convergence sequences or Cauchy.

The exact same proof will work word for word, but one interesting remark that can be made is the converse is not true. The converse is not always true. It is not true that if you have a Cauchy sequence in a metric space, then it converges; that is not necessarily true. A simple example should clarify what is happening.

Example: Just look at the set $(a, b) \subseteq \mathbb{R}$. Now, what we can do is we can treat (a, b) as a metric space; by considering the same absolute value function as the metric on (a, b). Then, it will be clear to you that the sequence $a + \frac{1}{n}$, n suitably large is Cauchy. In fact, $a + \frac{1}{n}$ must converge to a, but if you notice this a is not actually an element of (a, b). It is an element of $\mathbb{R}$, but our metric space itself does not have the point a. So, this is saying that you can have a Cauchy sequence that does not converge to any point in the metric space.

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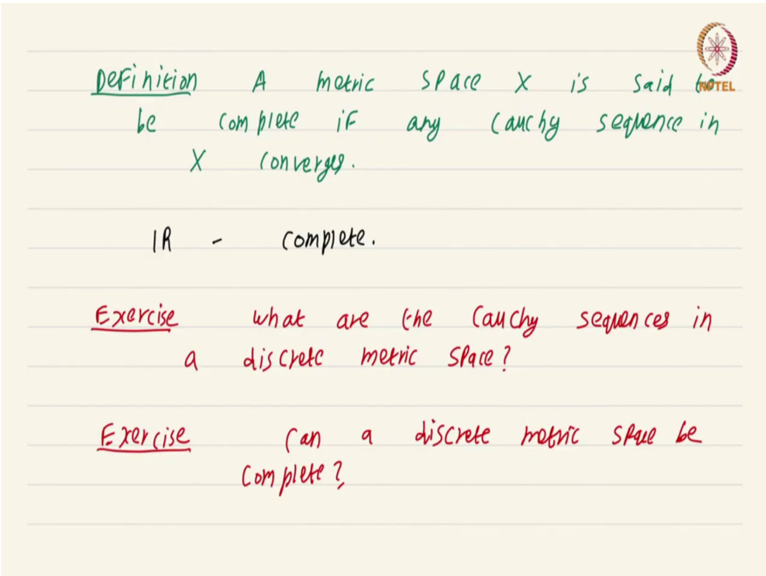

So, let me give one more definition in this regard.

Definition: A metric space X is said to be complete if any Cauchy sequence in X converges.

So, as you can probably guess, we have already proved this $\mathbb{R}$ with its usual absolute value as the metric is complete. This notion of completeness that we have just defined is equivalent to the completeness axiom that we spent a considerable amount of time studying in the context of ordered fields. So, this is just a remark that is not central to this course. Still, I just want to make that this notion of completeness, which is a sort of topological completeness, is equivalent to completeness in the sense of an ordered field that we define.

So, let us come back to this notion of completeness. $\mathbb{R}$ is a complete metric space. Work out these two exercises to understand this notion of completeness in greater depth.

Exercise: What are the Cauchy sequences in discrete metric space?

Exercise: Can a discrete metric space be complete?

Solve these two exercises then you will understand what exactly is going on with completeness. Onwards and forward another definition.

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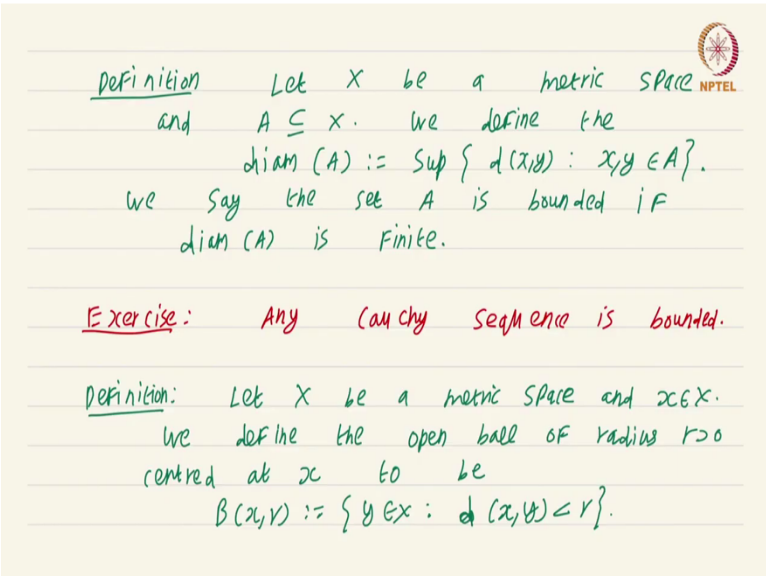

Definition: Let X be a metric space and $A \subseteq X$. We define

$$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}.$$

Look at all pairs of points that come from A, look at the distance between the pair of points and take the supremum. Now, the second part of the definition is more important.

We say the set A is bounded if the diameter of A is finite. So, whenever the diameter of a set is finite, we say that the set A is a bounded set.

Now, again another exercise for you. A plenty of things in this particular module will be left as exercise because, as I have mentioned repeatedly before, much of this set of definitions and basic theorems are just obtained by changing notation in the earlier chapter on the topology of real numbers.

Exercise: Any Cauchy sequence is bounded.

So, let me give one more definition and related to this notion of diameter definition.

Definition: Let X be a metric space and take a point x in X. We define the open ball of radius $r > 0$, centered at x to be,

$$B(x,r) = \{y \in X: d(x,y) < r\}.$$

The definition is completely self-explanatory, and again, it is more or less exactly the same as the definition of open balls that we have already seen for the real numbers.

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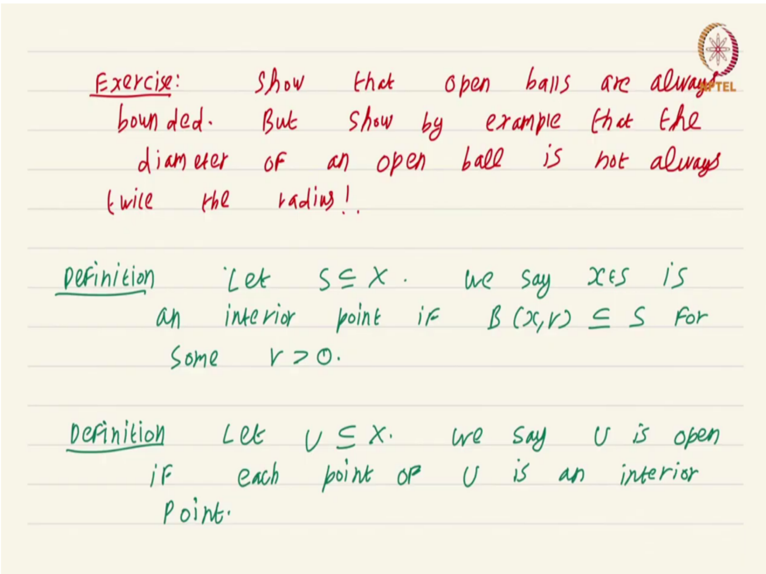

One exercise in this context is the following.

Exercise: Show that open balls are always bounded. It is rather a trivial exercise. You can even figure out what the bound is. But, show by example that the diameter of an open ball is not always twice the radius.

So, you will have to cook up a metric space in which it does not hold true that there is a ball, then automatically, the diameter of that ball is twice the radius of that ball. That is not going to happen in this particular metric space. Once we have defined open balls, the next definition should be fairly clear to you. I expect you to have guessed it.

Definition: Let X be a metric space. Let $S \subseteq X$. We say $x \in S$ is an interior if $B(x,r) \subseteq S$ for some $r > 0$.

And, the next definition is also going to be utterly self-explanatory.

Definition: Let $U \subseteq X$. We say U is open if each point of U is an interior point.

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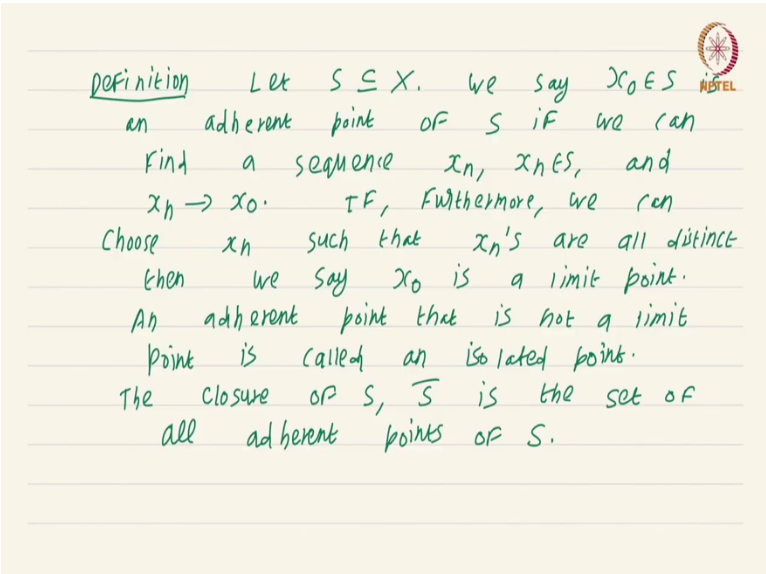

So far, so good all the definitions, as I had promised, are just changes in the earlier definitions and more to follow, which you should have anticipated again.

Definition: Let $S \subseteq X$. We say x_0 in S is an adherent point of S, if we can find a sequence x_n, such that $x_n \in S$ and x_n converges to x_0. If, furthermore, we can choose x_n such that x_n's are all distinct. This just means that there is no repetition in the sequence. Then we say x_0 is a limit point. An adherent point that is not a limit point is called an isolated point. An adherent point that is not a limit point is called an isolated point. And, the last definition is the closure of S, $\bar{S}$ is the set of all adherent points of S.

So, all these definitions are word for word the same. If you are getting bored, do not worry. We are just a few more definitions away, and then we can move onto more interesting material.

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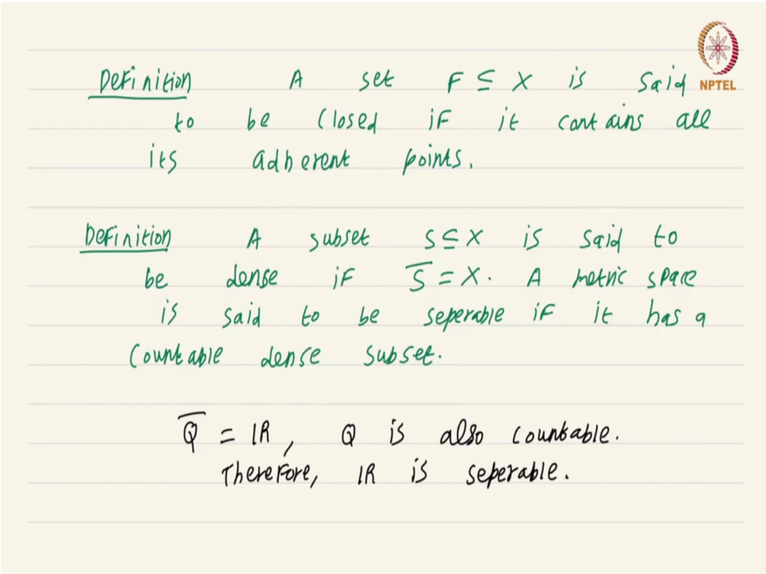

Yet another definition, this time that of a closed set.

Definition: A set $F \subseteq X$ is said to be closed if it contains all its adherent points.

So, just as a remark, I am not going to write this down. In the literature, what we call limit points are often called accumulation points or cluster points. And, again the terminology is not consistent across textbooks. What we have called an adherent point some authors call a limit point. So, it is a bit confusing. I strongly recommend that whenever you pick up a textbook that uses these concepts, always check for when the definition of the author what exactly the author means by a limit point or adherent point or accumulation point or cluster point. It is always a good idea to double-check.

The last few definitions in this extended module of definitions.

Definition: A subset $S \subseteq X$ is said to be dense if $\bar{S} = X$.

This is probably the only definition that is somewhat new. We have already seen it in some context, but this is probably the only definition, and this one and the next one to follow are the only definitions that are genuinely new.

Definition: A metric space is said to be separable if it has a countable dense subset.

So, a metric space is separable if you can find a subset that is both dense and countable. We already know that $\overline{\mathbb{Q}} = \mathbb{R}$, i.e., the closure of the rational numbers is the real numbers, and $\mathbb{Q}$ is also countable, which we saw long ago when we were still in kindergarten. So, we have seen that $\mathbb{Q}$ is also countable. Therefore, $\mathbb{R}$ is separable.

Now, the last definition in this module, you must have seen isomorphic vector spaces. You must have also proved that any finite-dimensional vector space is isomorphic to $\mathbb{R}^n$. This means that this isomorphism sort of acts like a dictionary that translates the elements in one vector space to that of the isomorphic vector space. So, isomorphic vector spaces are essentially the same vector space up to a renaming, and this renaming can be achieved by using an isomorphism. When are two metric spaces the same? Well, we have the following definition.

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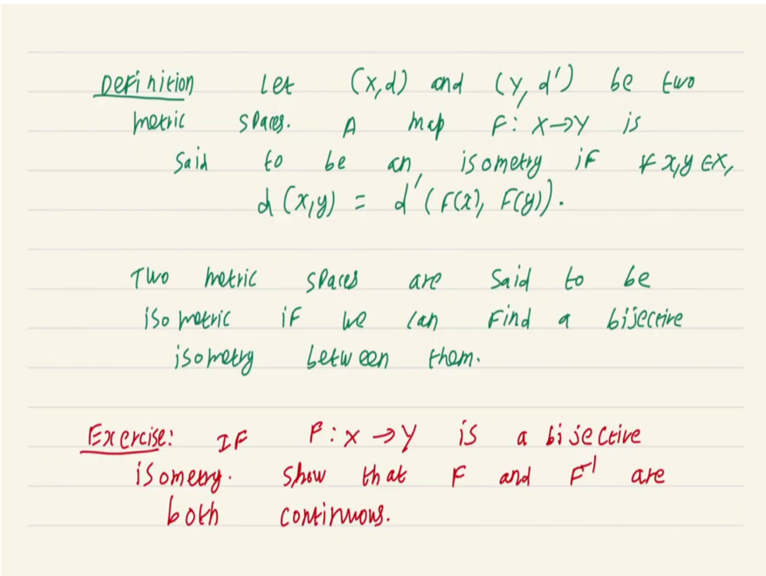

Definition: Let (X, d) and (Y, d') be two metric spaces. A map $f : X \to Y$ is said to be an isometry if for all pairs $x, y \in X$,

$$d(x, y) = d'\big(f(x), f(y)\big).$$

So, from the perspective of distances, the distance between a pair of points in the space X is exactly equal to the distance between the image points $f(x)$ and $f(y)$. Therefore, whatever holds for a pair of points in the space X, the same property will hold for the pair of points $f(x)$ and $f(y)$ in the space Y.

Definition: Two metric spaces are isometric if we can find a bijective isometry between them.

So, I am going to leave you with an exercise.

Exercise: Let X and Y are metric spaces. If $f : X \to Y$ is a bijective isometry, show that f and f^{-1} are both continuous.

So, this concludes an excessively long list of definitions. I thank you for your patience. This is a course on Real Analysis, and you have just watched the module entitle Loads of Definitions.