Real Analysis II
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Lecture - 11.2
Higher-Order Partial Derivatives

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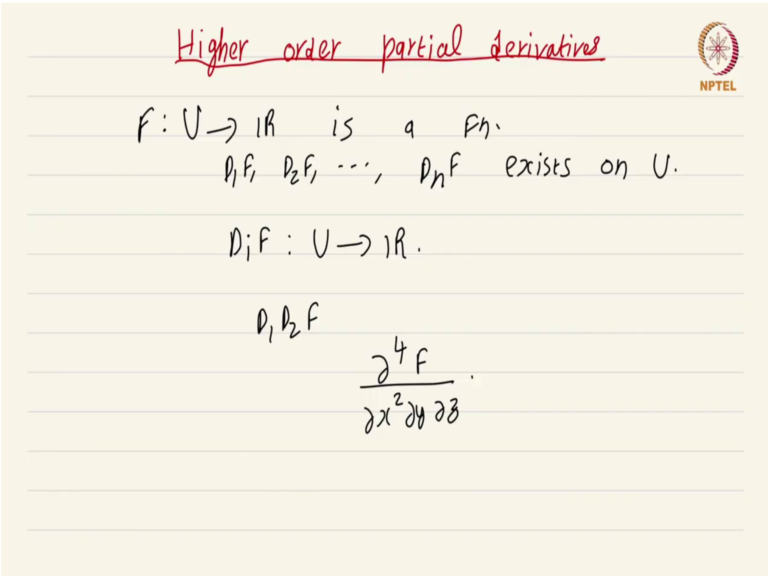

In this video I am going to be talking about Higher-Order Partial Derivatives. So, the setting is as always F from U to R is a function. Now, suppose the partial derivatives D 1 F D 2 F dot dot dot D n F exist on U.

So, this function is differentiable with respect to each of the variables throughout U. Then of course, I can consider this D i F has a function from 0 to R. So, I get new functions D 1, D 2, D 3 dot dot dot D n which are themselves function from the open set U taking values in R. Now, I can repeatedly take more partial derivatives.

For instance, I could take D 1 D 2 F this is exactly what the notation suggest. You first differentiate with respect to the second variable you will get a function D 2 F from U to R then you differentiate with respect to the first variable of this new function and in this way repeating this process you can get many combinations of derivatives.

In a standard multivariable calculus course at the undergraduate level you must have definitely come across horrible expressions that look something like this del 4 F by del x squared del y del z.

So, this is a typical expression that says you first differentiate with respect to z and you, then you differentiate with respect to y, then you differentiate with respect to x then again with respect to x. So, you I am going to assume that you are familiar with such notations and I am not going to use such horrible notation in this course we will use the more elegant multi index notation which will become very useful when we prove Taylor's theorem down the line.

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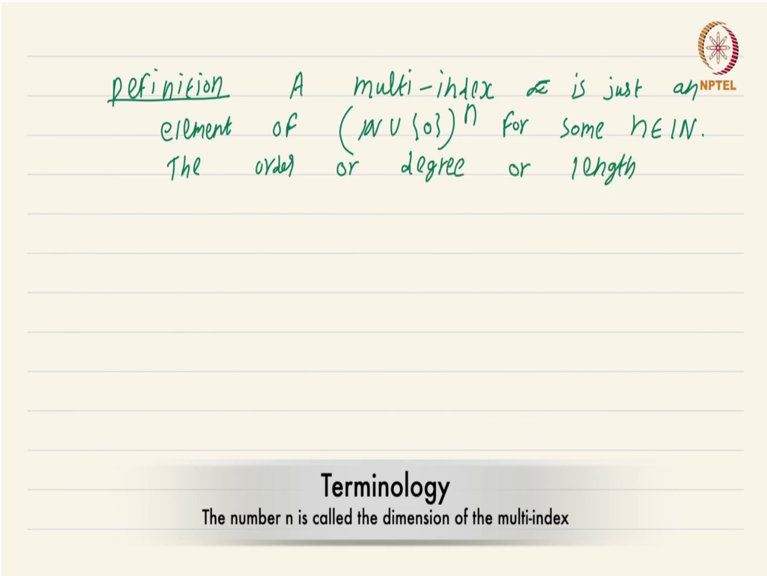

So, let me give the definition of multi index notation. It is a bit involved and it takes a bit of getting used to, but once you get used to it you will prefer this notation over the classical notation for partial derivatives. So, a multi index; a multi index alpha is just an element of the natural numbers union set with 0 Cartesian product with itself order n for some n for some n in the natural numbers. So, a multi index is just a tuple of non negative integers.

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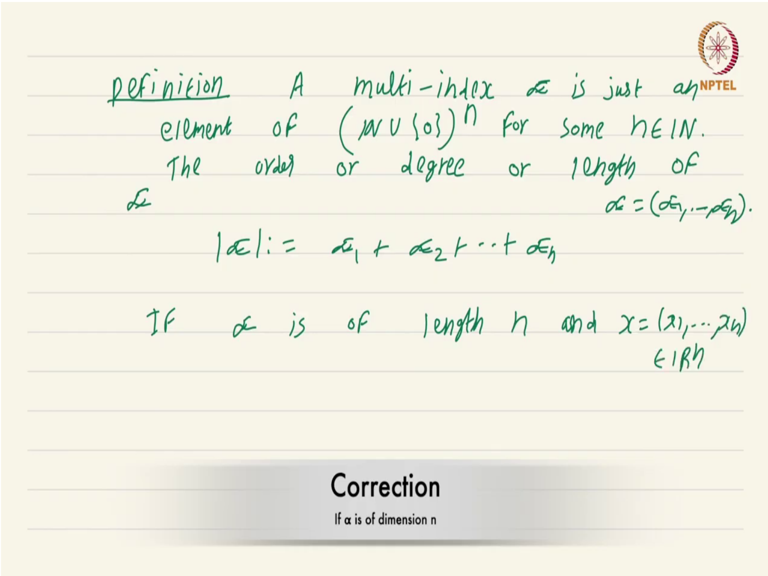

The order or degree or length of alpha this is denoted by mod alpha is just the sum of the various entries. It is just alpha 1 plus alpha 2 plus dot dot dot alpha n where alpha is equal to alpha 1 comma dot dot dot alpha n. So, if you take this tuple alpha 1 to alpha n and you want to find out the length you just take the sum. So, if alpha is of length n length n and x equal to x 1 to x n is an element of R n.

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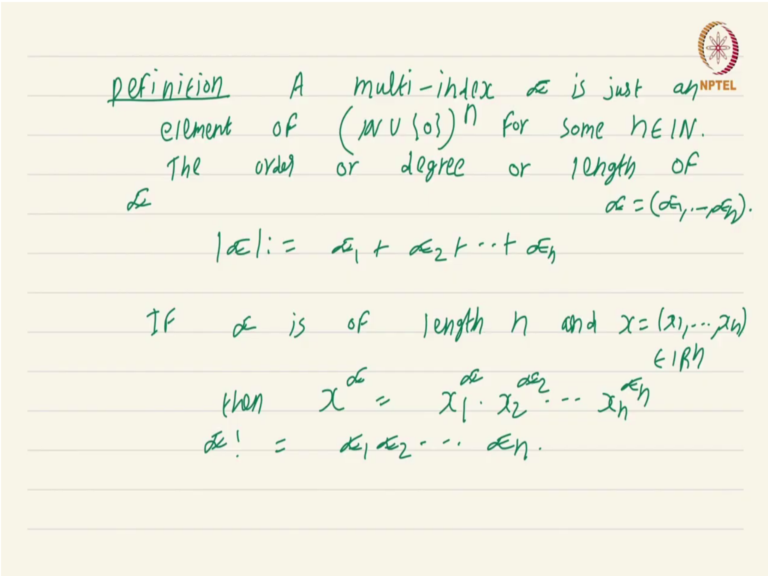

Then we define x power alpha to be x 1 power alpha 1 multiplied by x 2 power alpha 2 dot dot dot x n power alpha n and finally, we define alpha factorial to be alpha 1 alpha 2 dot dot dot alpha n ok. So, these are the various operations that you can do with multi indices. Of course, you can add multi indices. I am not going into that because it is just component wise fine. One more thing I need to define for you.

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If $F : U \to \mathbb{R}$ is a function then

$$\partial^{\alpha} F := \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

$|\alpha|$-th order partial derivative operator.

When the concerned partial derivative exists.

$$\frac{\partial^2 F}{\partial x_2 \partial x_1}.$$

If F from U to R is a function, is a function then we define del alpha F, the alpha-th partial derivative of F to be nothing but del mod alpha F by del x 1 power alpha 1 dot dot dot del x n power alpha n ok. So, of course, when the relevant partial derivative, when the concerned partial derivative exists, otherwise it makes no sense; when the concerned partial derivative exists, ok.

Now, so, such and del alpha we say is actually an alpha-th or sorry mod alpha-th order partial derivative operator ok. Now, observe that the way we have defined this partial derivatives using multi indices. There is no way to do del squared F by del x 2 del x 1. This is simply not possible, but this is not such a big deal because we will soon prove a theorem which says that the orders will not matter once you put a mild hypothesis on F.

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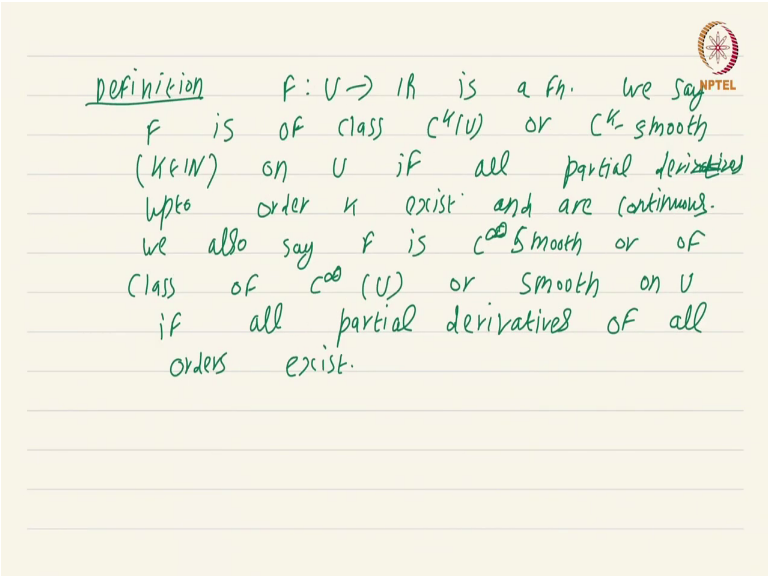

So, before I do that let me just make a very important definition which is going to be used throughout this course. This is the definition of smoothness classes. So, again F from U to R is a function, is a function we say F is of class C K U or C K smooth C K smooth. So, here K is a natural number, K is a natural number; C K smooth on U if all partial derivatives all partial derivatives up to order K exist and are continuous and are continuous ok. Then we say that the function is of class C K U or C K smooth, ok.

So, not only do you require the derivatives to exist, the derivatives must be continuous that is important, ok. We also say, we also say F is C infinity smooth or of class C infinity U or just plain smooth without any further qualifications or just smooth on U, if all partial derivatives, all partial derivatives of all orders of all orders exist. Note there is no continuity requirement here. The continuity requirement is automatic. I want you to prove that as a simple exercise, ok.

So, we coming back to the situation I described in the previous slide, where there is no way to express del squared F by del x 2 del x 1 in our concise notation involving multi indices. This is not much of an issue because in most scenarios the function will be well behaved enough so that these things are equal. So, that is a famous theorem which you must have learnt in multivariable calculus, but probably without proof.

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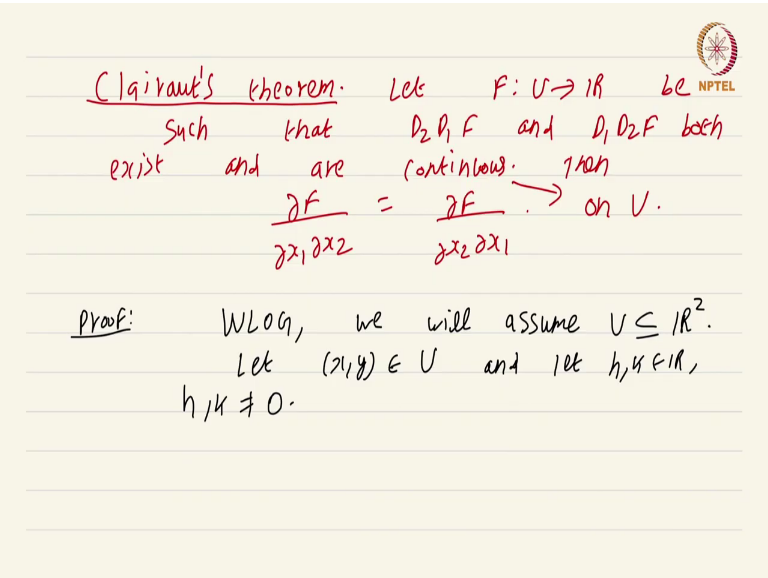

This is called Clairaut's theorem. So, the setting is again as follows. Let F from U to R be such that be such that D 2 D 1 F and D 1 D 2 F both exist and are continuous and are continuous. Then of course, I am going to write both are equal. So, I am going to write in that elaborate notation; del F by del x 1 del x 2 is equal to del F by del x 2 del x 1 both partial derivatives are equal, the second order partial derivatives are equal.

Now, there are weaker versions where you put enough conditions on one pair of partial derivatives to automatically ensure that the second pair of partial derivatives automatically exist. So, I am not going to prove those more general statements. This will be practically sufficient for all applications.

So, I am going to make a simplification and leave it to you to prove the general case. This is actually without loss of generality, but I am going to ask you to check why this is really without loss of generality. We will assume U is subset of, we will assume U is subset of R 2. So, we are only going to consider the two variable case, the general n variable case can be easily reduced to this situation which I am going to leave to you.

So, let the point x comma y be in U, ok. Of course, I did not mention it should have been obvious or continuous on the whole of U. So, let me just add it so that there is added clarity. So, take this point x comma y in U and let this another vector pair of numbers h comma K be not equal to 0 z are not equal to z. Actually this is a bad way of saying it. Let h comma K be in R, both h comma K are not 0 ok.

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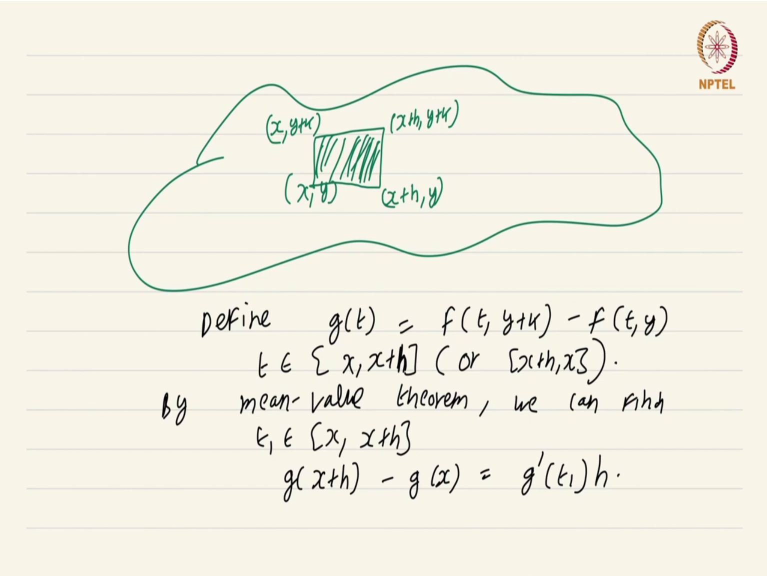

Now, so, let us draw a big picture. This picture will sort of guide what is happening. So, you have this domain U. I am focusing on this point y. What I am going to do is I am going to consider h and K so small that this entire rectangle, this entire rectangle this is x plus h, this is this is x plus h comma y, this is x plus h comma y plus K and this is x comma y plus K. So, I am going to choose h and K so small that this entire shaded rectangle in the in fact, the closure of the shaded rectangle is fully contained within the domain U.

So, I am going to only consider h and K so small, ok. Now, what I am going to do is I am going to compute the values of this function on the vertices of this rectangle I have just drawn. So, what you do is you define this new function g of t to be equal to F of t comma y plus K minus F of t y, ok. This is a one variable function and this is certainly defined whenever t is an element of x comma x plus h, ok.

Of course, I am treating h and K as positive. I want you to go I mean do the same arguments. It will it will be quite similar whenever h or K is negative. So, this actually should be or x plus h comma x, if h were to be negative this function g will be defined in the closed interval x plus h comma x. I am not going to treat the negative case. I am just going to consider the positive case and the proof in the negative case is exactly the same.

So, by mean value theorem, by mean value theorem; by mean value theorem we can find we can find t 1 in let us say x comma x plus h, it could be x plus h comma x that is really the only change you will have to make when adapting the proof to the negative case. We can find t 1 and x comma x plus h such that g of x plus h g of x plus h minus g of x is just g prime at t 1 times h.

We can apply the mean value theorem on this interval simply because we know that the partial derivative for D 2 D 1 and D 1 D 2 to exist both D 1 and D 2 must exist and they must automatically ok, I am not going to talk about continuity; both D 1 and D 2 must certainly exist and once D 1 and D 2 exist we can certainly apply this, ok. Here for instance only D 1 is needed, the existence of D 1 is needed, but the existence of D 2 will be quickly needed in the next few steps ok.

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$$F(x+h, y+K) - F(x+h, y) - F(x, y+K) + F(x, y)$$

So, what have we done? Expanding out this definition of g of t what we actually get is the following. F of x plus h comma y plus K y plus K minus F of x plus h comma y minus F of x comma y plus K plus F of x y, ok. So, this is the expanded out version of what I have done. Just wait a second. Let me just erase it. I think I wrote this wrong. Let us just do it the whole thing.

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$$g(x+h) = F(x+h, y+k) - F(x+h, y)$$

$$-g(x) = -F(x, y+k) + F(x, y)$$

So, when you substitute, when you substitute g of x plus h g of x plus h would be nothing but F of x plus h comma y plus K minus F of x plus h comma K F of x plus h sorry x plus h comma y not K right. And similarly, g of x would be just F of x comma y plus K minus F of x y and when you subtract this you will get a minus sign here and a plus sign here and then we can erase this. Yeah, I wrote it correctly not bad ok. I wrote it correctly.

So, you get F of x plus h comma y plus K minus F of x plus h comma y minus F of x comma y plus K plus F of x y.

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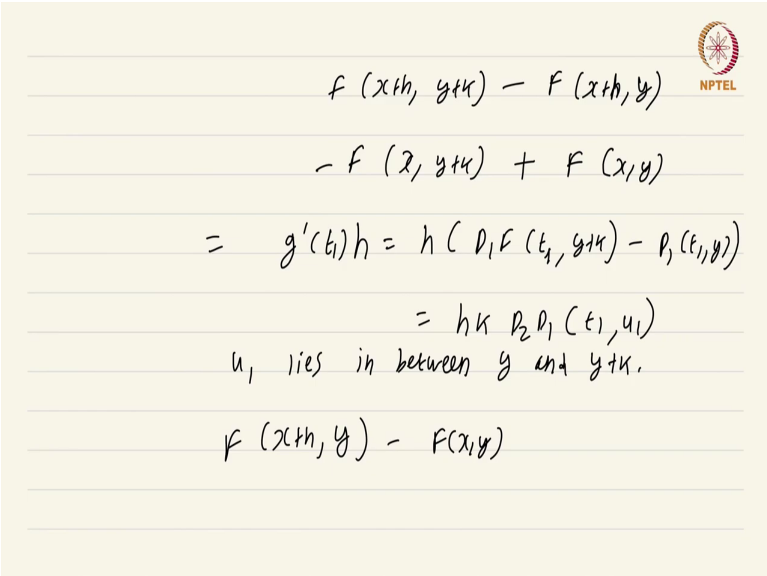

What does our result say this application of mean value theorem? This is just g prime of t 1 times h that is what we had got from the application of mean value theorem, ok. Now, expanding this out in terms of partial derivatives, this is just h of D 1 F of t 1 comma y plus K minus D 1 of t 1 comma y. This is just expanding out by partial derivatives.

Now, we have assumed that D 1 itself is differentiable in the sense that D 2 of D 1 does make sense. So, now, applying the mean value theorem to the function D 1, we get this is nothing but h K times D 2 D 1 t 1 comma u 1, where u 1 lies in between y and y plus K in, ok. Now, what we are going to do is we are going to compute this complicated expression F of x plus h comma y plus K minus F of x plus h comma y minus F of x comma y plus K plus F of x y in a different way; in a different way.

What we are going to do is we are going to consider the function the new function F of x plus h comma y minus F of x, y. In the first case we considered the function F of just a second. We considered the function g of t equal to F of t comma y plus K minus F of t y. So, this time we are going to consider F of x plus h comma t F of x plus h comma t minus F of x t.

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$$F(x+h,\ y+k) - F(x+h, y)$$

$$- F(x, y+k) + F(x,y)$$

$$= \quad g'(t_1)h = h\left(D_1F(t_1, y+k) - D_1(t_1,y)\right)$$

$$= hk\ D_2D_1F(t_1, u_1)$$

u_1 lies in between y and $y+k$.

$$F(x+h, t) - F(x,t)$$

$$hk\ D_1D_2F(t_2, u_2)$$

And we are going to apply the exact same argument we have done and we will be able to conclude that this big expression this big expression is in fact, equal to is equal to h K D 1 D 2 I here I forgotten F D 1 D 2 F of t 2 comma u 2, where again t 2 and u 2 are numbers in between x and x plus h and y and y plus K respectively ok.

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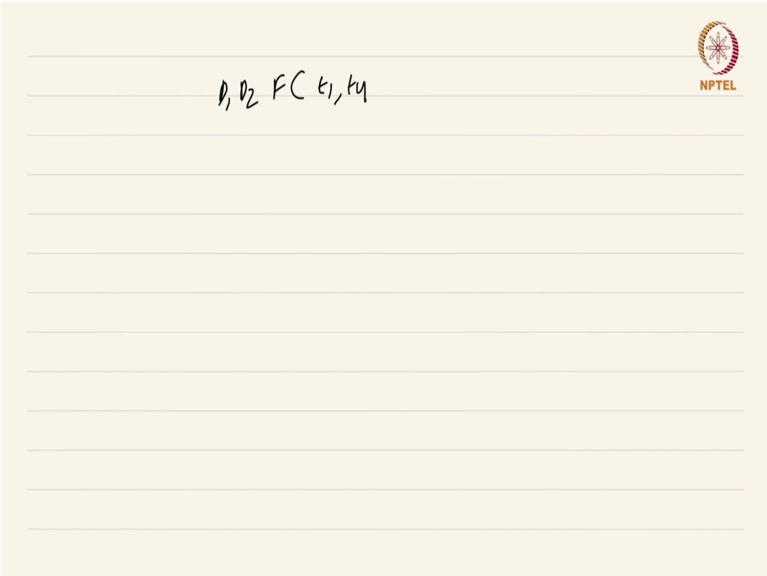

So, the net upshot is D 1 D 2 F of t 1 u 1 sorry.

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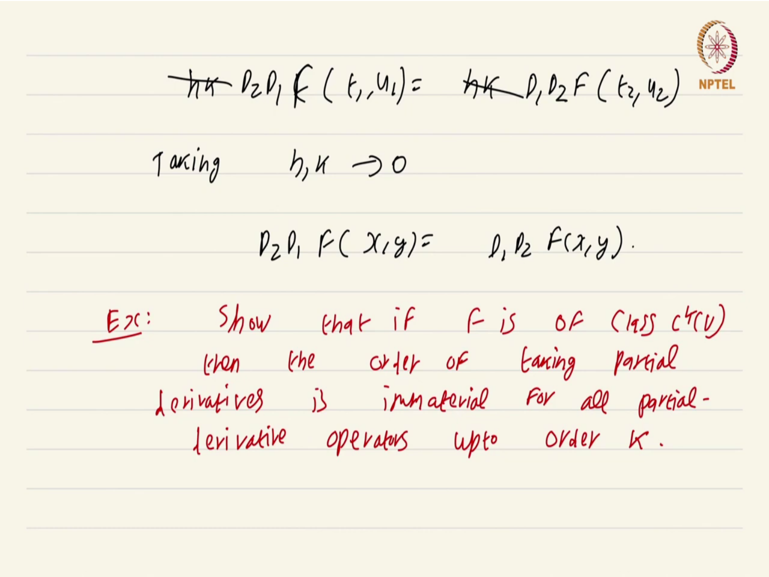

Let me go back and refer to the previous one yeah h K D 2 D 1 F of t 1 u 1 is equal to h K D 1 D 2 F of t 2 u 2. I hope I got that correct. I did not mess up anything yeah not bad. ok. So, h and K were assumed to be non negative. So, these two can be cancelled out and notice that this will yield t 1 u 1 t 2 e 2 irrespective of how small h and K is. If I shrink h and K I would get different values of t 1 u 1 in between x and x plus h and y and y plus K respectively, but this equality will be still true.

Taking h comma K to 0 both of them to 0 and by using the fact that the partial derivatives are continuous we immediately get that D 2 D 1 F of x, y is equal to D 1 D 2 F of x, y as claimed. So, the proof just involves taking the differences of the values of the function on the four vertices of a rectangle in different ways and applying the mean value theorem twice and then taking h comma K to 0 that is the basic idea of the proof.

So, I have solved it only in the two variable case. Now, I am going to leave a final exercise for you to finish. The exercise says show that if F is of class C K U then we can take we can take partial derivatives we can it does not matter, I will place it in a better way.

We can then the order of taking derivatives of taking partial derivatives, taking partial derivatives is immaterial for all partial derivative operators up to order K. It really does not matter. In the exercise in the notes I have given you an explicit example we are taking the order of partial derivatives will give you different result. This is a scenario where the hypothesis of Clairaut's theorem will not be satisfied.

This is a course on Real Analysis and you have just watched the video on Higher Order Partial Derivatives.