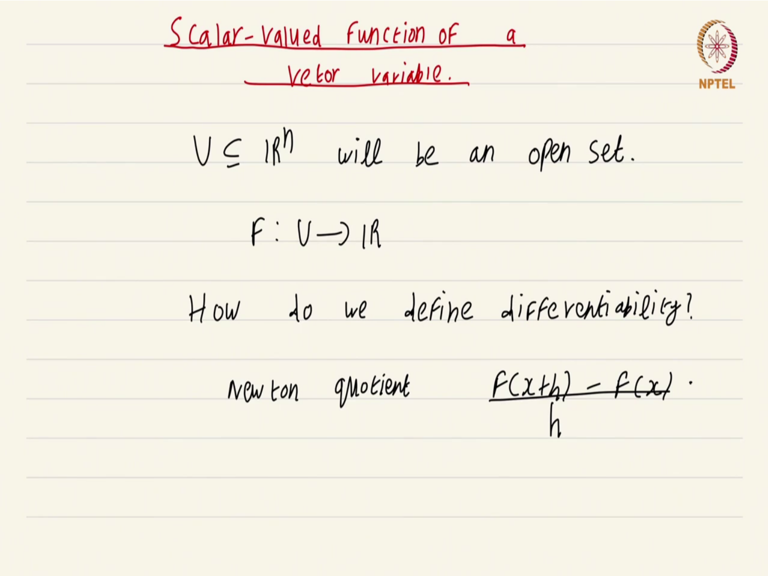
Real Analysis II
Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 10.2
Scalar-Valued Functions of a Vector Variable

(Refer Slide Time: 00:22)



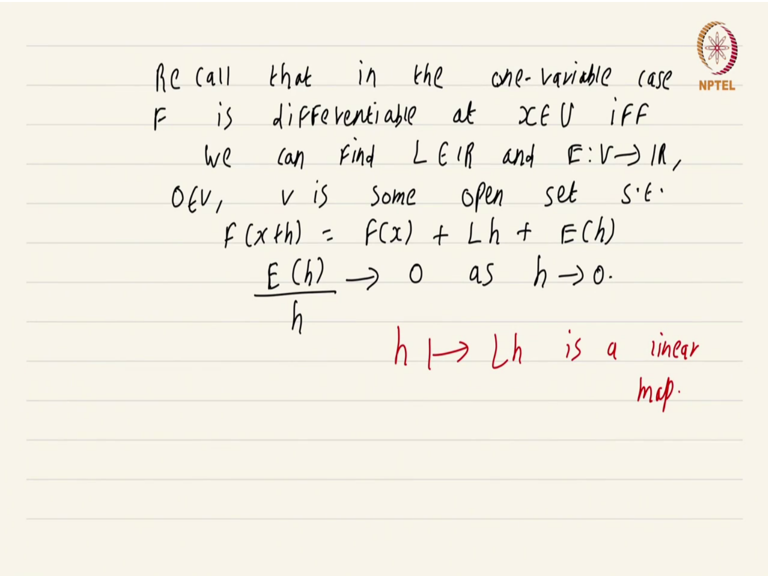
We move on to the next stage in our study that is Differentiating Scalar-Valued Functions of a Vector Variable. So, first let me make some notational remarks throughout U subset of R n will be an open set; will be an open set.

And I will not pre specify what the dimension n is, it is a fixed number greater than or equal to 1, whatever I am about to say also applies for the case n equal to 1. So, our aim is to now study a function F from U to R more precisely to study the differentiability of such a function.

Now, the first question that arises is how do we define, how do we define differentiability in this situation? One complication that arises is because we cannot divide by a vector. The natural thing to do would be to consider a Newton quotient just like what we did in the one-variable case, and consider something like this F of x plus h minus F of x by h.

Only issue is this is simply not possible in higher dimensions because h is now a vector. So, what do we do to fix the situation? Well, recall that when we define differentiability in the one-variable case, we had done a brief description of the derivative as the best linear approximation of the function.

(Refer Slide Time: 02:03)



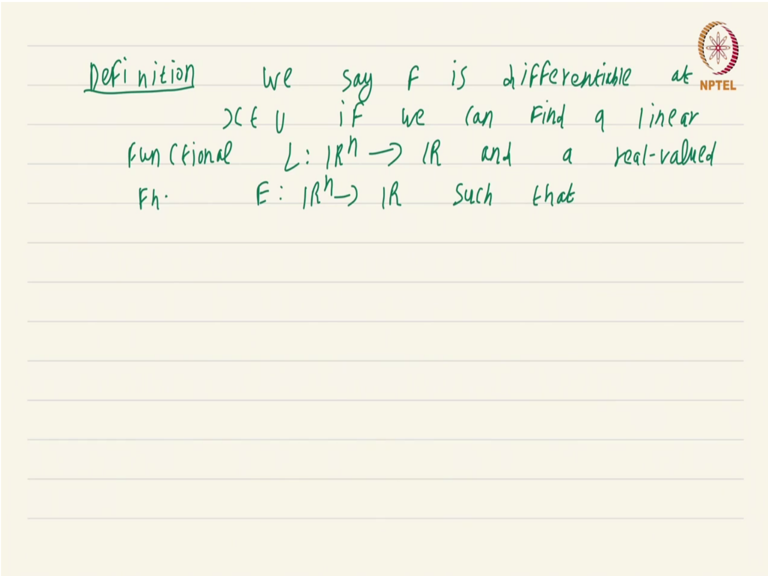
So, recall, that in the one-variable case one-variable case F is differentiable at x in U if and only if we can find we can find L a real number and E a function defined from V to R, 0 is in

V, and V is some open set, V is some open set such that F of x plus h is equal to F of x plus L h plus E of h. There we have E of h by h goes to 0 as h goes to 0.

So, this just says that if a function F is differentiable at the point x, then you can make a good linear approximation or rather more precisely an affine linear approximation, this part F of x plus L h is supposed to be the affine linear part, and this is an error term. And the error term is small.

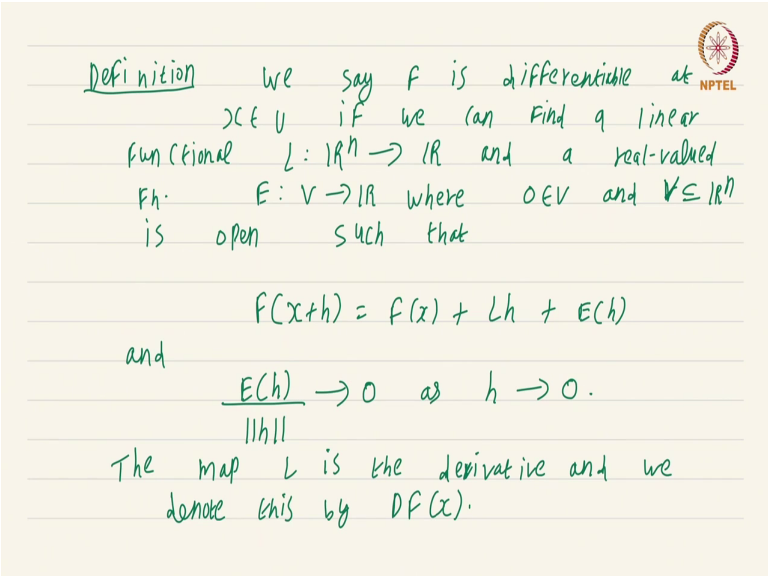
The smallness is quantified by the fact that E of h by h goes to 0 as h goes to 0 ok. Now, we are going to mimic this definition exactly. The key observation is the fact that the map h going to L h is a linear map in the one-variable case, multiplication by a scalar is a linear map in the one-variable case. So, this allows us this viewpoint of the derivative as the best linear approximation allows us to define the derivative of a scalar valued function of a vector variable. Let us do that.

(Refer Slide Time: 04:20)



Definition, as I mentioned throughout F from U to R will be a function, and U will be an open set. So, I am not going to write that in the definition. We say F is differentiable at the point x in U; the point x in U if we can find a linear functional L from R n to R linear functional is just another specific term for a linear transformation but taking values in the base field which happens to be R ok, and a real valued function real valued function E from R n to R such that actually the E is not from R n to R.

(Refer Slide Time: 05:38)



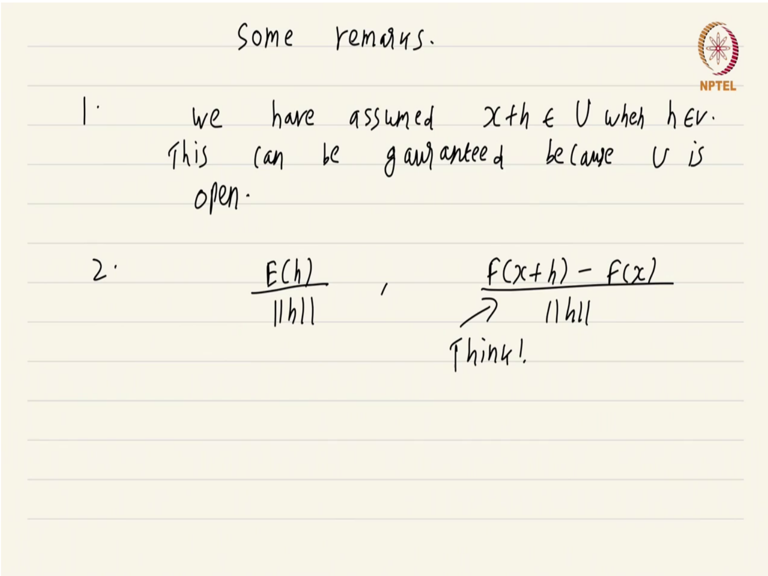
E is from E is from V to R, where this V is a neighborhood of 0, where 0 is in V, and V subset of R n is open such that F of x plus h, F of x plus h is equal to F of x plus L h plus E h. And E of h we cannot divide by the vector h, but we can take the norm E of h by norm h goes to 0 as h goes to 0 ok.

So, the first question that should come to your mind is what is this norm in the denominator. Well, it really does not matter. All norms on R n are the same, but whenever I consider the Euclidean space R n I am going to be considering the Euclidean norm. And similarly I am always going to consider only the Euclidean metric on R n ok. So, you divide by norm h, and this should go to 0 ok, E of h by norm h should go to 0.

Now, if you look back at the definition on the last page which I have written here in the one-variable case, and look at this definition here, both are identical. So, we have just taken another formulation of the derivative in one-variable and just declared that same expression to be defining the derivative in higher dimensions.

So, the map L is the derivative, the map L is the derivative is the derivative. And we denote this, and we denote this by DF x, we denote this by DF of x ok. So, the derivative in this scenario is modeled on our observation that the derivative in one-variable is the best linear approximation.

(Refer Slide Time: 08:14)

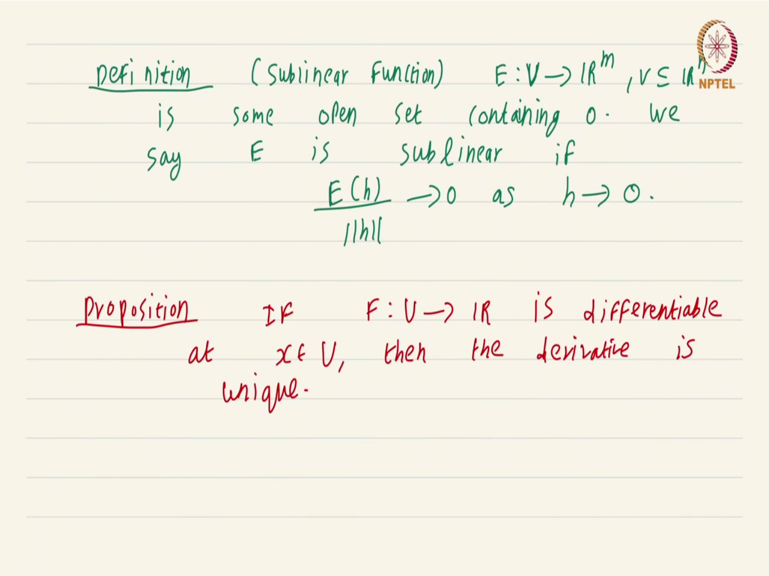


So, let me just make some remarks. So, the first remark is we have assumed we have assumed x plus h is in U right. So, recall that E the function E is defined on some open set that contains 0 ok. We are assuming that x plus h belongs to U, so that you can actually take F the F on that. So, this just I means x plus h belongs to U when h is in V. This can be guaranteed can be guaranteed because U is open, U is open.

So, h is defined on some pre specified open set V, we can shrink that V if necessary to ensure that x plus h is always an element of U whenever h comes from V this is not such a major issue. So, the 2nd remark is more interesting. Now, when we define the derivative, we are considering quotients of this type E of h by norm h because we are unable to divide by vectors.

So, why did we do this complicated stuff about linear approximations and all? You could have just considered this Newton quotient F of x plus h minus F of x by norm h, and taken h to 0. What would happen in this scenario? Would this give the derivative, will this give anything meaningful? I want you to think about this. Why did not we just define the derivative using this modified Newton quotient? Ok.

(Refer Slide Time: 10:07)



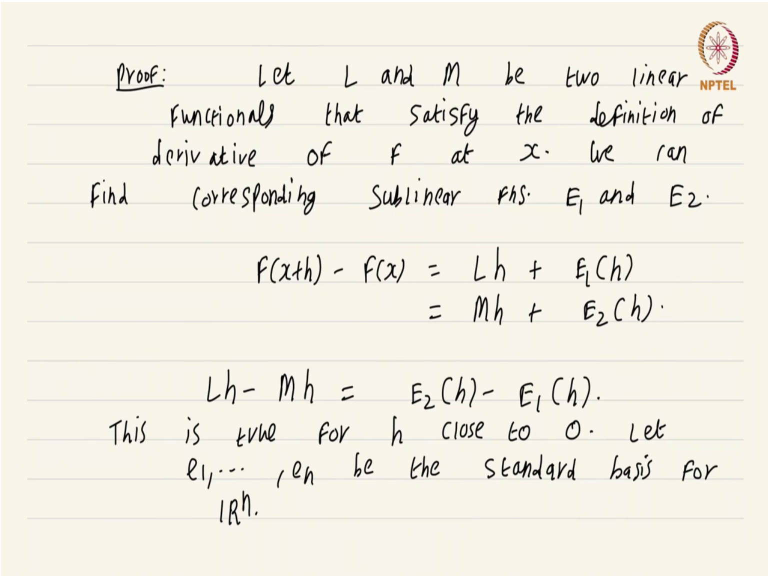
Let me make one more definition because we are going to be dealing with this error function frequently in the proofs. And we have to discuss the behavior of the error function as h goes to 0. Let us make a convenient definition for this. This is the definition of a sublinear function.

The definition goes as follows. Again E is from U or rather V to R m this time it really is I am not requiring it to be taking values only in the real numbers, where V subset of R n is some open set that contains 0. It, I do not really care some open set containing 0 ok. We say E sublinear, we say E is sublinear if as you can expect E of h by norm h goes to 0 as h goes to 0 ok.

So, we are going to use this notion of sublinear offered in proofs where we have to deal with these error functions. And I mean this the having a convenient expression for such functions will simplify the proofs.

So, we immediately start with the first proposition. And this is an important proposition to prove though the proof is rather trivial is the fact that if F from U to R is differentiable at x in U at x in U, then the derivative is unique. You cannot have two different expressions parading as the derivative ok.

(Refer Slide Time: 12:17)



So, the proof of this is rather easy, proof. Let L and M be two linear functionals that satisfy the definition of the derivative definition of derivative of F at x ok. So, we can find we can find corresponding functions, corresponding sublinear functions E 1 and E 2 ok. What this means is that F of x plus h minus F of x is equal to L h plus E 1 h where E 1 is going to be sublinear, it is also equal to M h plus E 2 h ok.

This just means that L h minus M h is equal to E 2 h minus E 1 h rather straightforward algebra. Now, this is true, this is true, for h close to 0, right. Why close to 0, because we want x plus h to be in U right. Now let e 1, to e n be the standard basis the standard basis for R n. We want to show that L e i equal to M e i that will show that L equal to m because a linear functional is completely determined by its behavior on a standard basis.

(Refer Slide Time: 14:53)

Set $h = te_1$, $t > 0$ suitably small.

$$L te_1 - M te_1 = E_2(te_1) - E_1(te_1).$$

$$Le_1 - Me_1 = \frac{E_2(te_1)}{t} - \frac{E_1(te_1)}{t}.$$

$$\frac{E_2(te_1)}{\|te_1\|} \to 0 \text{ as } t \to 0 \qquad 0$$

As $t \to 0$ $\quad \dfrac{E_2(te_1)}{t} \to 0$

$$Le_1 - Me_1 = 0. \qquad Le_i = Me_i, \quad L = M.$$

Now, x plus E 1, x plus E 2, and so on may not be in U. So, what we do is V scale. Set h to b t e 1 t greater than 0 suitably small. How small, so that x plus t e 1 is actually going to be in the open set U ok. Now, observe that you have L t e 1 minus M t e 1 is equal to E 2 of t e 1 minus E 1 of t e 1 that is just the last line where I have substituted the specific value for h ok.

Now, since t is not equal to 0, we can divide both sides by t. And because L and m are both linear, I can take this t outside in both right. So, dividing by t, I will get L e 1 minus m e 1 is E 2 t e 1 by t minus E 1 t e 1 by t, right. Now, the left hand side is independent of t. So, when you take t going to 0, the left hand side is going to remain unaffected the limit will exist and the limit will be equal to L e 1 minus M e 1.

What about the right hand side? Well, let us just look at one term just focus on this. We know that E 2 t e 1 by norm t e 1 goes to 0 as t goes to 0, but since we are in the Euclidean setting and E 1 is a standard basis vector norm E 1 is just 1. So, this just immediately gives that as t goes to 0 as t goes to 0, E 2 t e 1 by t goes to 0. And the same remark applies for the second term. This term also goes to 0 as t goes to 0.

Net upshot is L e 1 minus M e 1 is 0, there was nothing special about E 1. The same proof works for the other standard basis. So, in fact, this concludes the proof that L e i equal to M e i which just means that L is equal to M because a linear functional is determined entirely by its behavior on a basis ok. So, the derivative is unique.

So, let us see some examples of differentiation. We will see more in the next video where we talk about the gradient, but for the time being let us set the stage and consider just at least two examples.

(Refer Slide Time: 17:42)

1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is constant. Then f is differentiable at all points $x \in \mathbb{R}^n$ and $Df(x) = 0$ - functional.

$$f(x+h) - f(x) = 0.$$

2. If f is a linear functional then f is differentiable at all points of $\mathbb{R}^n$ and $Df(x) = f$.

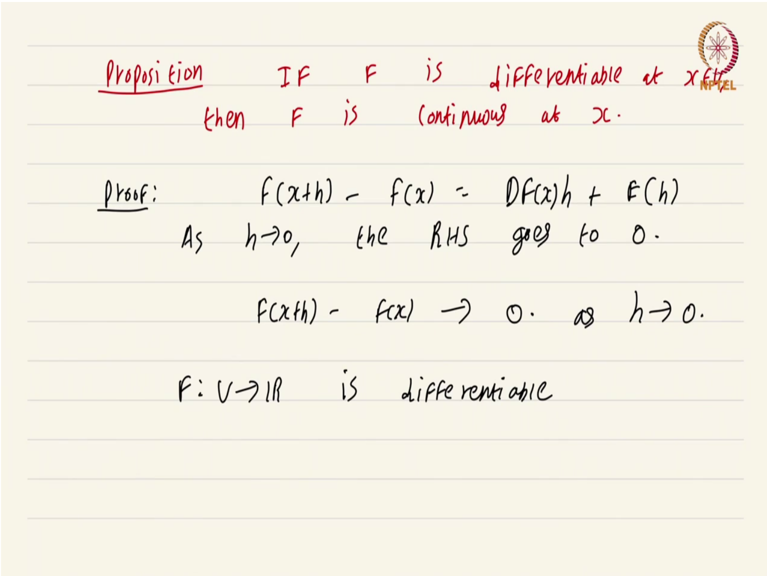
$$f(x+h) - f(x) = f(x) + f(h) - f(x) = f(h)$$

Suppose F is constant, suppose F from R n to R is constant. Then F is differentiable, F is differentiable at all points x in R, x in R n of course, sorry all points x in R n, and as you can guess D F x is going to be 0 is the 0 functional. Why is this? Well, this should be obvious to you, but you can just write down and see this also F of x plus h is F of x plus h minus F of x is just 0. So, we can just take the 0 functional and the error function to be also the 0 function, and we would be done ok.

Second example is equally trivial, but somewhat more interesting. If F is a linear functional, if F itself is a linear functional, then F is differentiable at all points of R n and DF x is just F ok. Now, this is again obvious because F of x plus h minus F of x is just F of x plus F of h minus F of x which is just F of h. So, we can just take the linear functional F in the definition of the derivative and take it as the candidate derivative, and take E to be 0 again.

So, in this case also F is differentiable everywhere and the derivative is F itself which is sort of to be expected because the derivative is supposed to be the best linear approximation. And since F is already linear you do not need to approximate F ok. So, we will conclude this short video by one more proposition which is also easy.

(Refer Slide Time: 20:12)



Proposition, if F is differentiable at x in U, then F is continuous at x. Well, how do you prove this? Proof, well, we know that F of x plus h minus F of x is equal to this DF x h plus E of h ok. Now, as h goes to 0, the right hand side, the right hand side goes to 0. Well, the first term DF x h obviously goes to 0, DF x is a linear mapping and the term E of h goes to 0 even after you divide by norm h and take h to 0.

So, without that norm h, it is going to go to 0 even faster ok. So, as h goes to 0, F of x plus h minus F of x goes to 0 ok. So, this concludes the proof that the function F is continuous at the point x. So, one final remark. So, in the future when we say F from U to R is differentiable, we will say such expressions, what this means is that F is differentiable at every single point of the domain U ok.

So, again the definition of the function being differentiable at an open in an open set just means it is true point wise ok. Now, the next video is going to be about the gradient and an explicit way to compute the derivative. So far this definition is not very useful in computation. The definition just tells you the characteristic property of the derivative.

It does not tell you how to find out what this derivative is going to be. The next video, will fix that. We will find an explicit way of computing the derivative using partial derivatives. This is a course on Real Analysis, and you have just watched the video on Scalar-Valued Functions of a Vector Variable.