Real Analysis II
Prof. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 10.1
Vector-Valued Functions

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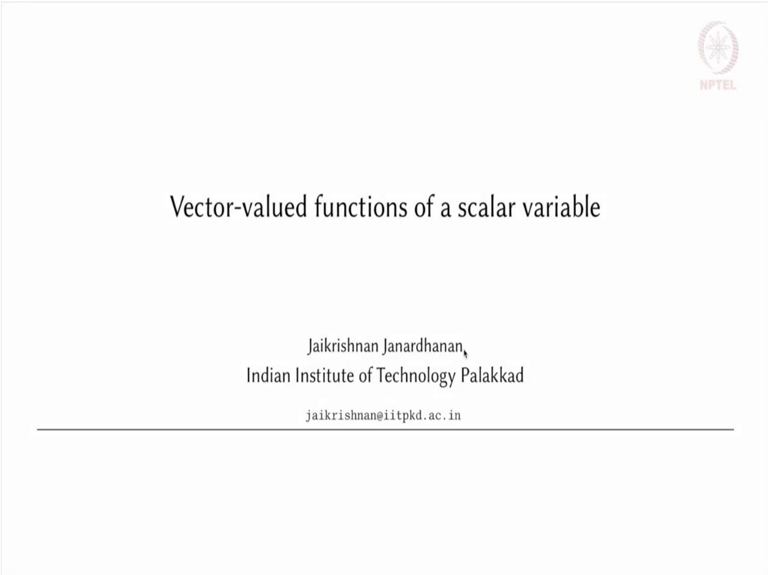

Now, study how to define the derivative for a Vector-Valued Function of a scalar variable, in other words of curves.

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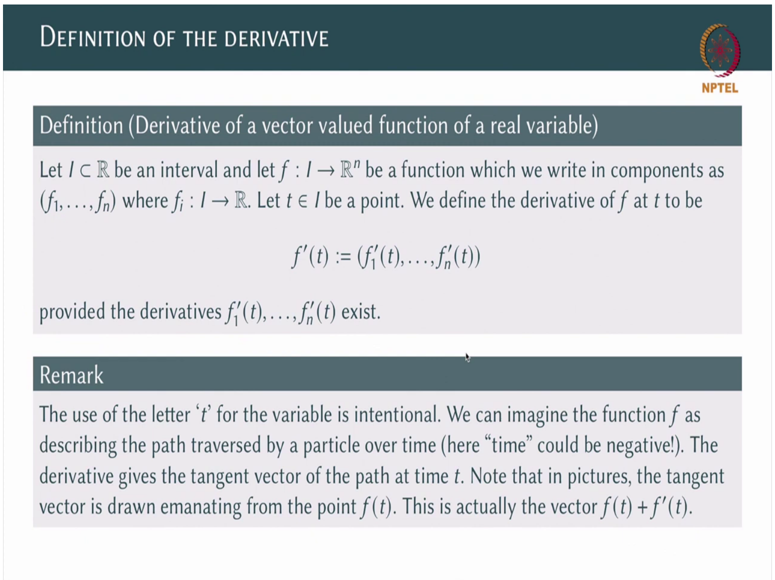

Now, the definition of the derivative in this scenario is rather straightforward. So, throughout this part we will be considering intervals I, which are subset of R, and we are considering a function f from I to R n. Now, since this is a function taking values in R n we can write down the function in terms of its component functions f 1 to f n, where each f i is a function from I to r.

Now, fix a point t in I, we want to define the derivative of the function f at this point t. We define the derivative to be quite simply the vector whose component functions are nothing, but f 1 prime of t, f 2 prime of t, dot dot dot f n prime of t. So, to differentiate a vector-valued function of a scalar variable just differentiate each one of its components. Of course, this definition makes sense only when the derivatives f 1 prime dot dot dot f n prime exist.

So, this is as you can expect an utterly straightforward extension of the usual notion of the derivative to a more general setting, and most of the facts about this scenario is going to be quite straightforward and is going to be left to you as exercises. So, one remark I would like to make, usually we use the variable x, but here I have used the variable t. This is intentional.

This serves a psychological aid, that these functions f of a scalar variable taking vector values this these are supposed to be curves. So, we can imagine the function f as describing the path traversed by a particle over time. Of course, here we have the weirdness the time could even be negative. I leave it up to you to imagine what this could mean.

The derivative gives the tangent vector of the path at time t, this is should be familiar to you from a basic course on multivariable calculus. Of course, when you draw pictures you draw the tangent vector as emanating from the point f of t. This is not actually accurate. When you just write down a vector you actually mean that the tail of the vector is at the origin.

So, to be 100 percent precise, if you want to write down the I mean if you want to draw the picture that you are you have no doubt seen in elementary books on multi-variable calculus. The vector that you usually call the tangent vector is f of t plus f prime of t. So the vector f prime of t which is the derivative has been translated to the point f of t.

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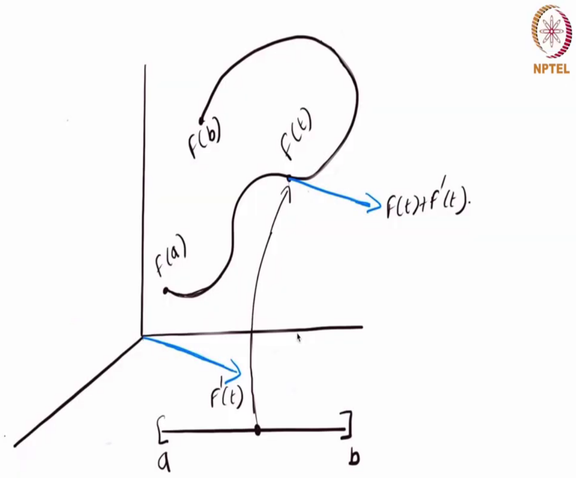

Figure: Curve in $\mathbb{R}^3$

So, here is a picture that illustrates what is going on. You have this particle that is moving in space like this, ok. At the starting point, it is at F of a, at the ending point it is at F of b. We are focusing on one particular point t, it takes the value F of t, and I have drawn somewhat approximately what that tangent vector F prime of t is going to be. Of course, the real F prime of t will be based at the origin, ok.

The way I have drawn it is not exactly accurate. I have just parallelly moved the thing to the origin, but because this is a three-dimensional picture you must imagine this vector in an appropriate way. So, this is the actual vector F of t plus F prime of t. So, this is a very rough and actually somewhat inaccurate picture of what is going on, but it will convey the basic idea to you, ok.

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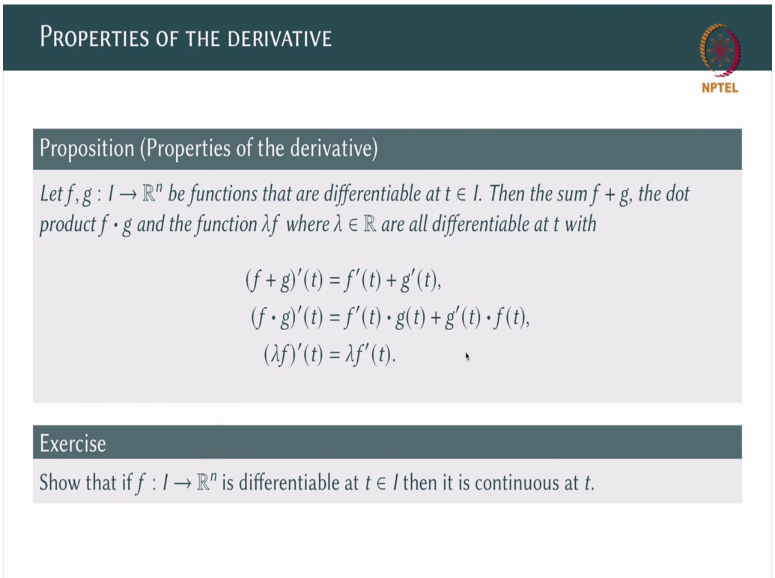

Since, everything is just doing things term wise, you can expect that all the familiar properties of the derivative will be true. So, suppose you have two functions f comma g from I to R n, let me recall once again that I will always denote a interval in R.

Suppose, you have these two functions f comma g that are differentiable at the point t. Then the sum f plus g, the dot product f dot g, which I want you to recall from your elementary studies in under graduation lambda f, where lambda is a scalar.

These are all differentiable at t with the following properties. The derivative of f plus g is nothing, but the derivative of f plus the derivative of g, both taken at the point t. Note, these two are vectors, so I can obviously, add them. The dot product f dot g that will give you a scalar valued function if I think about it. So, this is actually f dot g is going to be a regular one variable function taking values in R.

The derivative of that is nothing, but f prime of t dot product g of t plus g prime of t dot product f of t. And the third property is rather the easiest to prove. The derivative of lambda f is just nothing, but lambda f prime at t. So, these properties are rather easy and I am going to leave it to you to check.

Now, I am going to give you one more exercise and again this exercise is going to be rather easy. Show that if f from I to R n is differentiable, then the function is continuous at t. Now, it is at this point you must ask me, ok I want me to show it is continuous, but what is the metric on R n. Recall, from our studies from metric spaces that it really does not matter which one you put, which metric you put on R n provided it satisfies the basic property that sequences converge if and only if the component sequences converge it really would not matter.

But, when I do not mention what the metric on R n is you must always take the Euclidean metric which we have studied in some detail when we studied metric spaces. So, you under the Euclidean metric on R n, please show that any differentiable function is automatically continuous.

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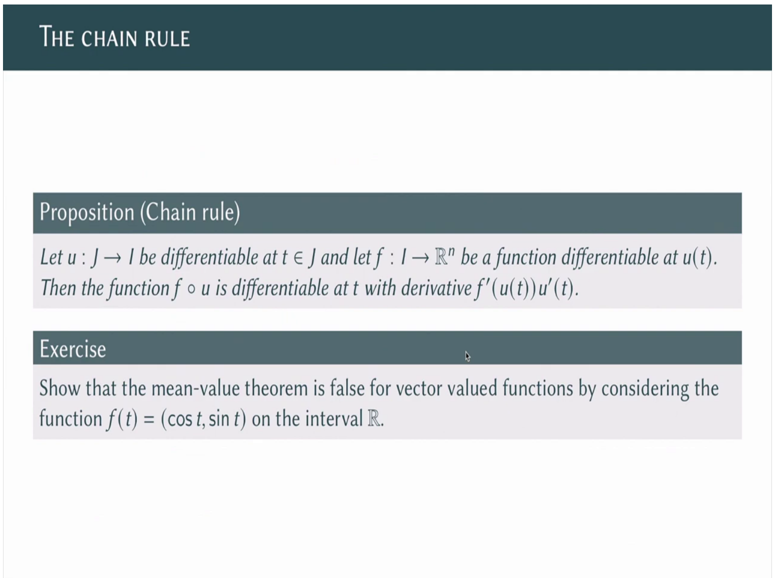

We also have a chain rule for such functions. Suppose, you start with an interval J, J is also an interval in R. Suppose, u from J to I is differentiable at the point t in J and you have a function f from I to R n that is differentiable at the point u of t. Then, the composite function, f composed with u is differentiable at t with the derivative f prime of u of t into u prime of t, ok.

So, note, u prime of t would be a number, f prime of u of t would be a vector. So, this is a number times a vector which is a vector which is what is to be expected because f composed with u starts at J and ends with R n. So, it is a vector-valued function of a scalar variable. Again, I am not going to prove this. This is again left for you as an exercise, as an easy exercise if I might have.

Now, so far I have said that everything goes through in an easy way. This is all extensions of the familiar facts from calculus. But one fact is not true, the mean value theorem in its direct formulation is actually false and I want you to work this exercise out. Show that the mean value theorem is false for vector-valued functions by considering the function f of t equal to cos t sin t on the interval R.

I am not going to say anything more as to what exactly mean I mean by the mean value theorem because that will spoil the surprise. Please work out this exercise its very interesting. So, this concludes this basic video on Vector-Valued Functions of a Scalar Variable and you are watching this course on Real Analysis.