Real Analysis II
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Lecture - 9.2
All Norms on a FDNVS are Equivalent

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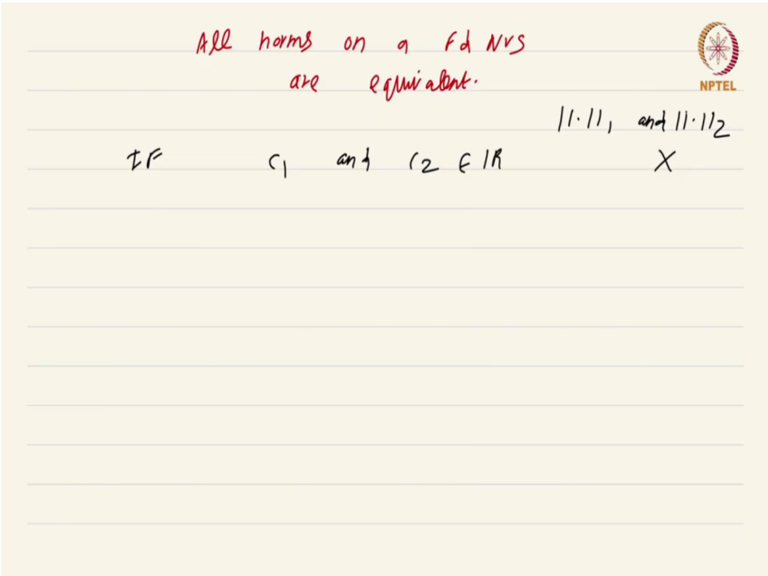

In this final video on metric spaces we are going to show a nice fact that all norms on a finite dimensional norm vector space are equivalent. Recall that 2 metric are considered equivalent if you can find constants c 1 and c 2 if you call the metric d 1 and d 2 you canfind constants c 1 and c 2. Such that c 1 d 1 of x y is less than or equal to d 2 of x y is less than or equal to c 2 d 1 of x y ok.

And this should be true for all x comma y inthe metric space xnot in the metric space in the set x. Now, you can show that 2 norms will generate equivalent metrics if you can find constants c 1 and c 2 calling the norms norm 1 and norm 2 on the set X on the vector space X, you can find constants c 1 and c 2 in the real numbersor rather c 1 and c 2 greater than 0.

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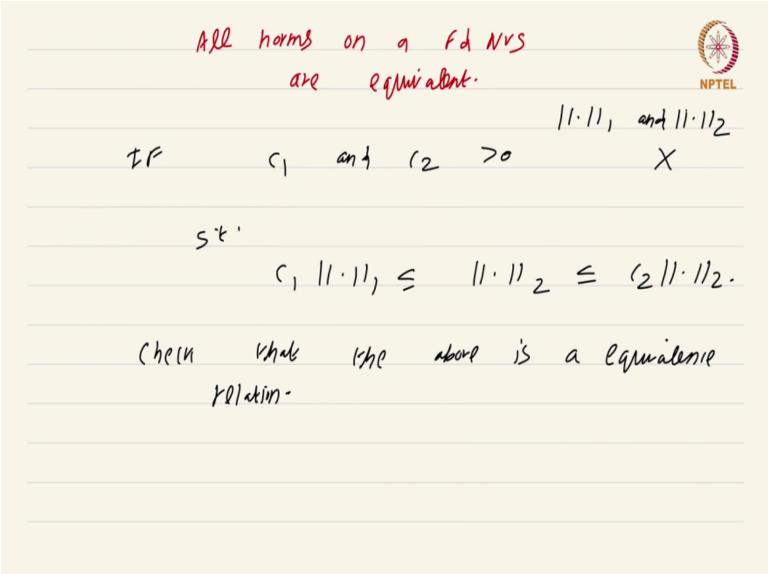

Such that c 1 norm 1 is less than or equal to norm 2 is less than or equal to c 2 norm 2. If you can do this then thesetwo norms will generate or will give rise to the same metric space andnot same metric space equivalent metric spaces and open sets everything will be same in thesetwometric spaces. Now, first of all I am going to leave it to you to check that the above is an equivalence relation insulation above is a equivalence relation.

If soyou look at the collection of all norms on a given finite dimensional vector space, here for this part finite dimensionality is not that essentialgiven a normed given a vector space look at the collection of all norms on that vector space and put an equivalence relation saying thattwonorms are equivalent if you can find constants c 1 and c 2 such that this happens.

Then show that this is an equivalence relation on the set of norms ok. So, once you have checked that let me now proceed and give you a quick sketch of a proof that all norms on a finite dimensional normed vector space are equivalent ok.

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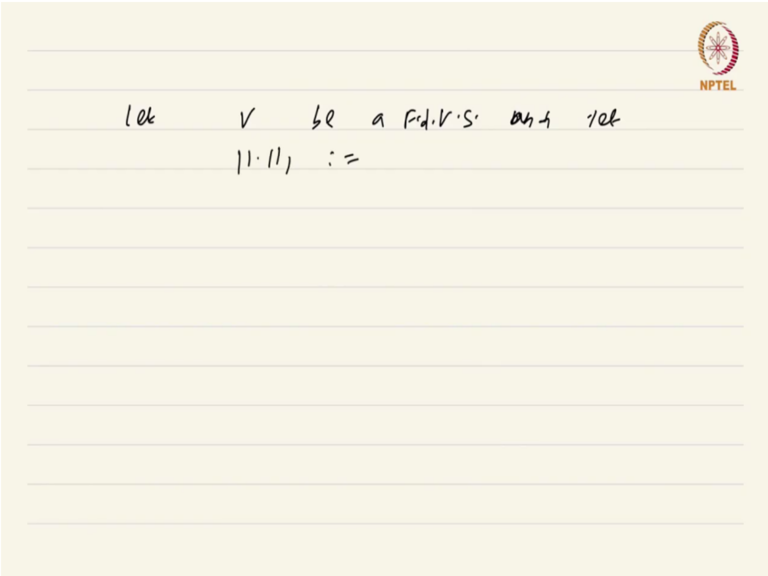

So, let vbe a vector space finite dimensional. So, finite dimensional vector space let norm 1 be by definition. So, before that what I am going to do is I am going to use this equivalence to show that all norms are equivalent to a single norm that we start out with.

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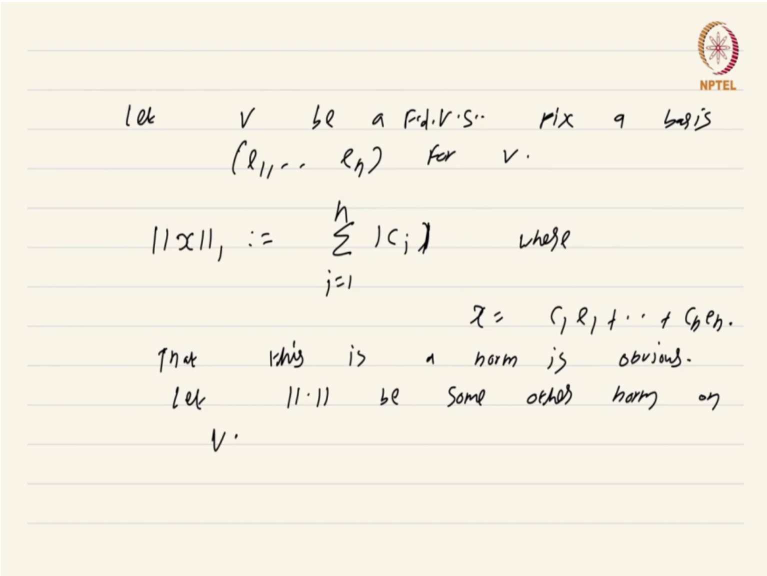

So, that norm is defined as follows fix a basis a basis e 1 to e n for v, we define norm of x 1 by definition to be summation mod c i i running from 1 to n. Where x in it is coordinates with respect to the base c 1 to e n is written as x equal to c 1 e 1 plus dot dot dot c n e n ok.

So, the idea is very simpleyou fix a basis e 1 to e n then you define the norm using this basis as the sum of the absolute values of the coordinates, that this is a norm is trivial and you have already seen this norm before in the context of r n. That this is a norm is obvious please do the simple checks and confirm for yourself that this is indeed a norm ok. Now, let norm be some other norm some other norm on v.

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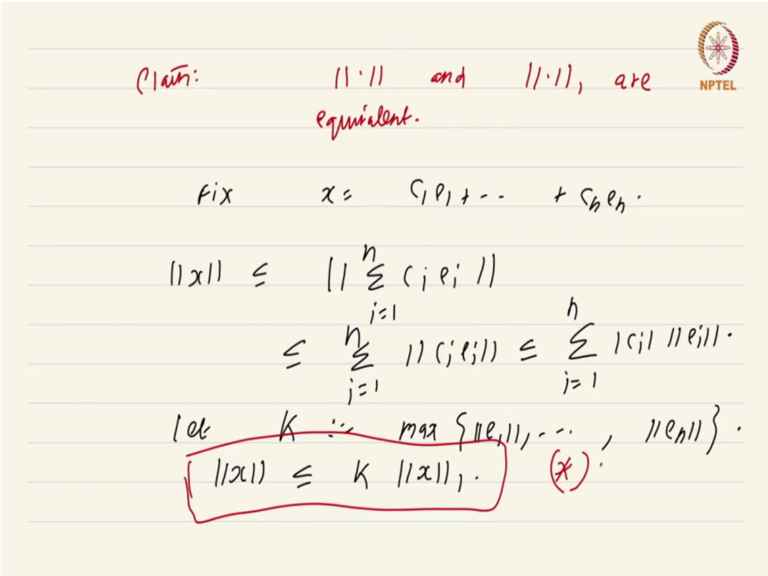

Now the claim is that norm and norm 1 are equivalent ok.How does one show this? Well let us just use the basic properties of the norm to see if we get something. So, look at a vector x sofix x equal to c 1 e 1 plus dot dot dot c n e n then norm x is less than or equal to, since I need to find constants c 1 and c 2 ok its not a problem there is going to be a repetition of constants, but that is not such a big deal.

So, this is going to be norm of summation c i e i i running from 1 to n and by triangle inequality this is less than summation i equals 1to n norm c i e i, which is less than or equal tosummation i equals 1 to n mod c i norm e i ok.

Now let this capital Kconstant be the by definition maximumof norm e 1 comma dot dotdot norm e n look at the maximum of this. So, the net upshot is norm x is less than or equal to this constant Ktimes norm x 1 ok. So, one direction of the inequality we have got we have got that norm x is less than or equal to Ktimes norm x 1 if you could.

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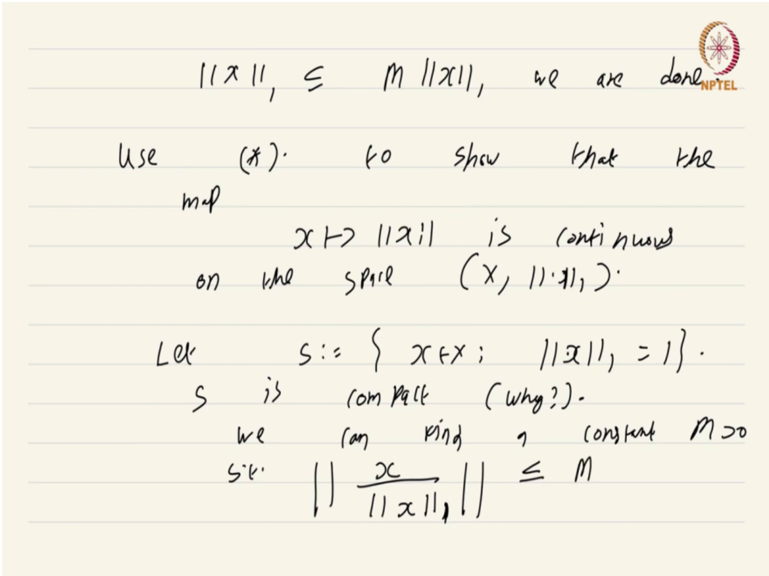

Now, show somehow that norm x 1 is less than or equal to is less than or equal to some other constant. Let us say m times norm x we are done we are done ok. Now before we proceed with showing that norm x 1 is less than or equal to m norm x, what we do is we look at a consequence of the factthat norm x is less than or equal to k times norm x 1 ok.

So, what I am going to ask you to do is use star which is this I have highlighted this equation and called it star, use star to show that the map x going to norm x is continuous on the space on the space Xcomma norm x 1. So, what you do is you put the metric coming from norm 1 on x put the usual Euclidean metric on r then the map x going to norm x is actually continuous ok.

Now how is this relevant well look at the set S which is defined to be x in xsuch that norm x 1 is less than or equal to 1. This is the closed unit ballor actually you can just take equal to 1 also you donot need to take less than or equal to 1this is the closed unit sphere under norm 1. So, consider this then S is compact then S is compact.Why? I want you to see why is S compact hint you can use the Heine Borel theorem in some way to show that S is actually compact think about how ok.

Now, because S is compact and the function x going to norm x is continuous we can find we can find a constant capital M greater than0, such that x divided bynorm x1 the norm of this quantity the norm of this quantity is less than or equal to capital M ok,or rather greater than or equal to capital M; sorry about that this is greater than or equal to capital M.

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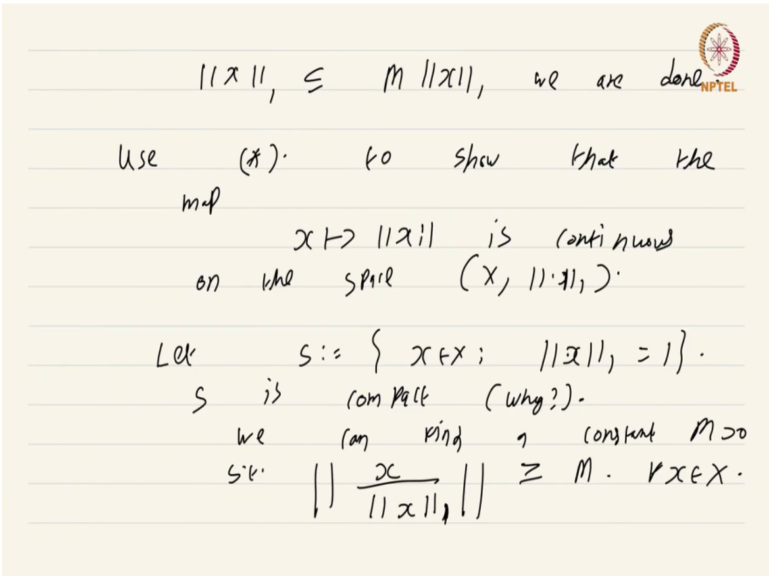

Let us see what is happening this quantity x by norm x 1 is actually an element of S. So, since S is compact the function norm on S attains both its maximum and minimum.

So, this capital M is essentially the minimum of this function. So, norm of x by norm x 1 is greater than or equal to Mwe can find a constant M such that this is true for all x in Xright. This just follows from the fact that continuous functions on a compactset attain maxima and minima ok. And also this we arecrucially using the fact that this minimum cannot be 0, because I am saying Mis greater than 0these are details that I want you to check.

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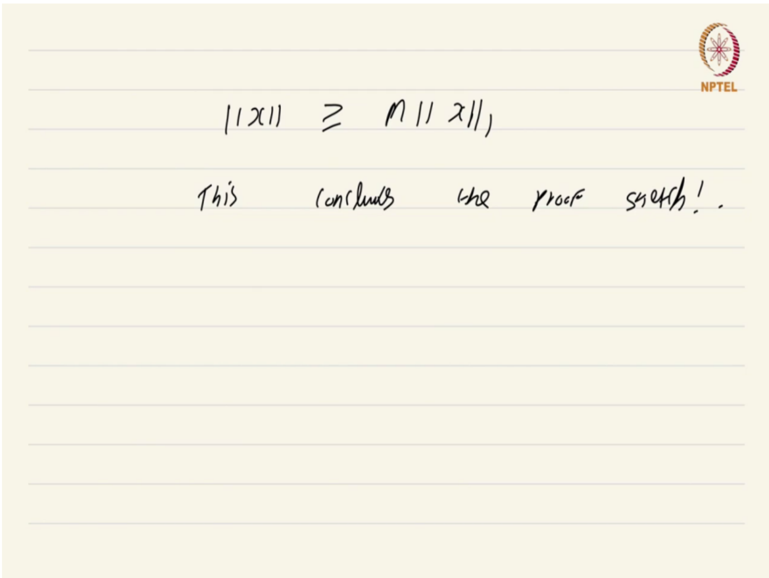

Now, what is this shows, this shows that norm x is greater than or equal to Mtimes norm x 1 ok. Which is exactly what is here which is exactly what is here this concludes the proof this concludes the proof or rather it is just a sketch. So, many steps are left for you so many steps are left for you, so please check all the details.

So, the basic idea is quite simple you fix one norm this norm x 1 which is really basic and try to show that any arbitrary norm, norm x is actually equivalent to this, then by the fact that this is an equivalence relationyou are done ok. So, this shows that all norms are equivalent on a finite dimensional vector space in particular on r n also all norms are equivalent.

So, in the rest of the course when we are studying differentiation in r n it really wouldnot matter which norm you put on r n, it will all be equivalent and we shall exploit this quite a lot in the proofs that is to come. This is a course on real analysis and you have just watched the video on all norms are equivalent.