Real Analysis II
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Lecture - 9.1
The Stone Weierstass Theorem

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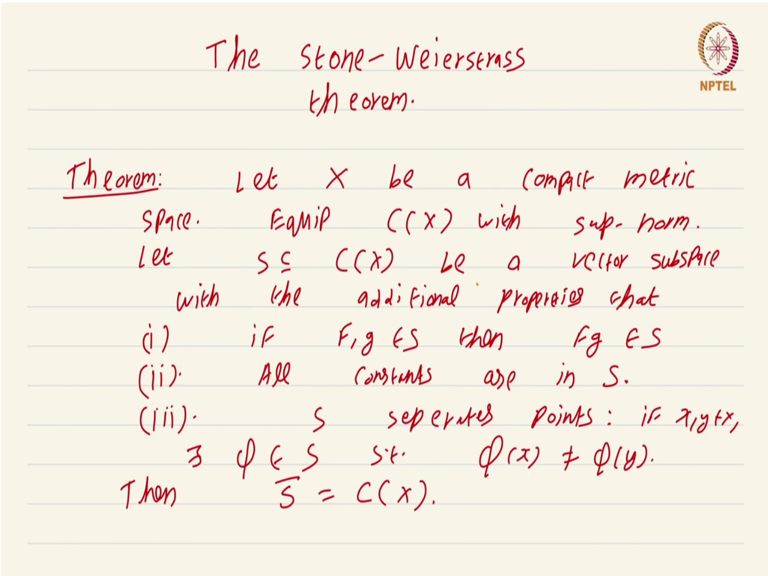

In this video we are going to study my favorite theorem from Real Analysis. Recall that the classical Weierstrass theorem says that any continuous function on a closed and bounded interval in r can be approximated uniformly by polynomials in real analysis one we have seen a proof of this using Bernstein polynomials.

Now, we are going to see a more abstract and general version of the Stone-Weierstrass theorem that not only greatly generalizes the classical Weierstrass theorem, but it also gives a new perspective into the classical Weierstrass theorem. The proof of the Stone-Weierstrass theorem will require a basic version of the classical Weierstrass theorem in one step, but apart from that the proof is completely self-contained.

So, that one step if you could give a self-contained proof, this Stone-Weierstrass theorem at one shot gives a proof of a very general type of Weierstrass theorem. So, without further ado let me just state the theorem a bit of terminology is involved I will try to keep it very basic. So, here is the theorem let X be a compact metric space. So, as you can see from a closed and bounded that is a compact interval in r we are jumping all the way to a compact metric space.

So, let X be a compact metric space equip C X the space of continuous real valued functions on x with the sup norm with the sup norm. So, this is our favorite metric space let S subset of C X be a vector subspace. So, we are considering the collection of all continuous and continuous functions on X and we are considering a vector subspace s; that means, the sum of two functions in S is in S and lambda times F is in S if lambda is a real number so on and so forth.

With the additional property; with the additional property that if F comma g are in S then F g is in S ok. So, let me just list out I need three different properties. So, with additional properties let us make it properties with the additional properties that number 1, if F comma g are in S then the product F g is in S the product of two continuous functions is continuous. So, we are requiring this vector subspace of C x to also be closed under multiplication.

In fancier language this is essentially going to make S into a sub algebra, S will be both the ring as well as a vector space and the operations between scalar multiplication and the product will behave well.

So, I am not going to state the more general definition of an algebra you can look it up in any standard textbook or Wikipedia ok. So, the product of two functions is also in S. Number 2, all constants are in S constants are in S any constant function is automatically continuous I am requiring this sub algebra S to contain all the constants.

And number 3, this is a crucial property S separates points what do I mean by this? If x comma y are in x, there exit a function phi in S such that phi of x is not equal to phi of y. No matter what pair of points x comma y in you choose in x you can always find a continuous function phi in S such that phi of x is not equal to phi of y.

Now, under these conditions the conclusion is then S closure is equal to x not x a C of X is C of X. So, what this is saying in more down to earth language is any continuous function on a compact metric space can be approximated uniformly by functions coming from S. Now the first order of business is for you to pause the video and check that indeed the set of polynomials on a closed and bounded interval indeed satisfy all these three properties.

So, in some sense that is a sanity check to check that this Stone-Weierstrass theorem in fact, generalizes the classical Weierstrass theorem ok. So, we have to show under these abstract conditions that S closure is equal to C of X and the proof of this is not hard and its extremely elegant we proceed step by step. So, let us go on with the proof; let us go on with the proof.

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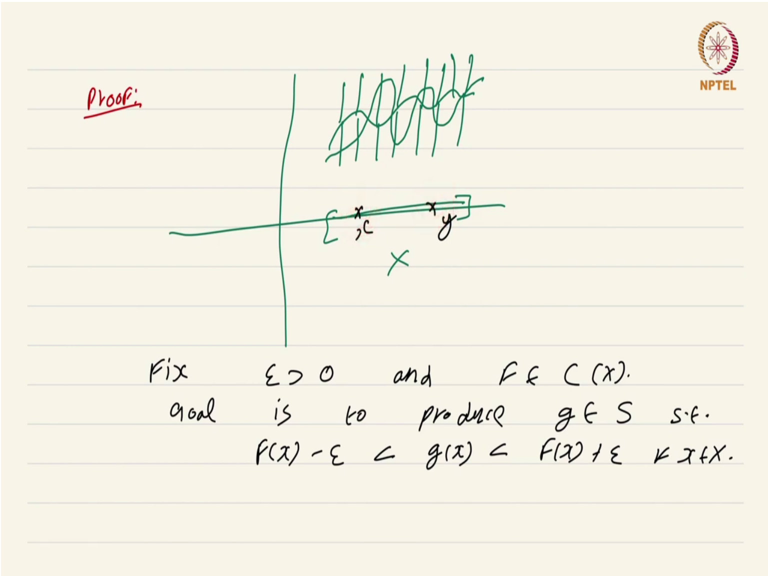

So, I am going to draw a bad picture as usual, but this bad picture is going to serve as a nice intuitive guide towards the proof. So, I am going to draw the x, y plane even though this set x could be anything and let us say for simplicity sake we are going to take x to look like this and we have some continuous function ok.

Now what is it that we have to do? So, this is what we have to do fix epsilon greater than 0 and F in C x goal is to produce g in S such that F of x minus epsilon is less than g of x is less than F of x plus epsilon for all x in X.

You agree with me what we have to do is we have to look at this epsilon with neighborhood of this function something like this, something like this. In this epsilon width we have to find some other continuous function that lies entirely within this epsilon width and this epsilon is completely arbitrary. For a fixed epsilon we have to find a function g which is squeezed in between F of x minus epsilon and F of x plus epsilon.

So, we are going to approach this by divide and conquer let us first try to show that we can find a function g that satisfies this g of x is less than F of x plus epsilon ok. Now what I am going to do is, I am going to make an additional assumption and I will sort of highlight how to remove this additional assumption towards the end of the proof I will give a sketch and you complete the proof.

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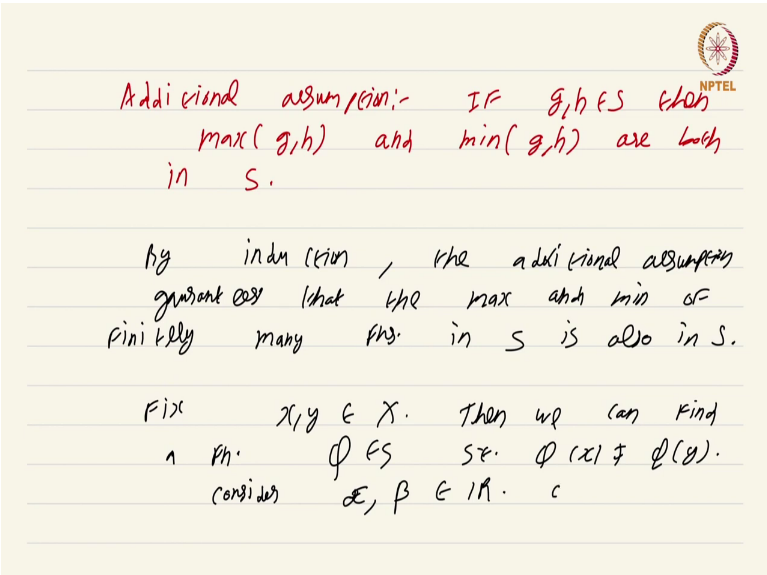

So, what we are going to do is we are going to make an additional assumption which is key additional assumption. The additional assumption is as follows if g comma h belongs to S, then max of g comma h and minimum of g comma h are both in S note that the maximum and minimum of two continuous functions is automatically continuous if you cannot see it immediately please pause the video spend 5 to 10 minutes and figure out why the maximum and minimum of two continuous functions is always continuous.

So, we are going to make this additional assumption about S and towards the end of the proof I will highlight how we can get rid of this additional assumption. In fact, it is this getting rid of this additional assumption that actually requires the classical Weierstrass approximation theorem ok.

So, I will just briefly say that by induction the additional assumption guarantees that the max and min of finitely many functions; finitely many functions in S is also in S. So, you can show that the maximum and minimum of finitely many continuous functions is continuous, what we are requiring of this sub algebra S is that it is not only closed under multiplication and addition, but its also closed under taking maximum and minimum finitely many times.

So, this second part this by induction is rather trivial you can just skip it its so, easy now let us proceed to the original proof. We want to show that given a fixed epsilon and a function F in C X we can squeeze a g in between like this.

Now, what is the only thing that asserts the existence of functions about our algebra S? If you notice the first condition does not say anything about existence, it just says that given two functions F and g you can combine them and that will also be in S we know that all constants are in S which is a good start we also know that S separates points ok.

So, the fact that S separates points tells us that there are plenty of continuous functions. In fact, it says that given any pair of points x comma y in S there is at least one continuous function that separates them so, that sort of generates for you a number of continuous functions on S and we are going to exploit the existence of these separating functions.

So, what you do is the following fix x comma y in x, then we can find; we can find a function phi of x not phi of x phi in S; phi in S such that phi of x is not equal to phi of y this is just rewriting I am writing the entire hypothesis again.

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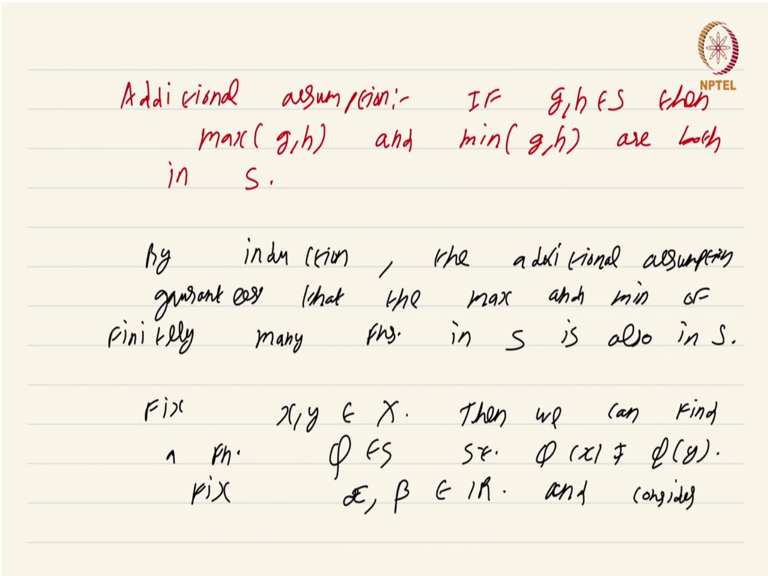

Now, what you do is, consider alpha comma beta real numbers fix two real numbers alpha and beta or rather let me just say fix alpha comma beta fix alpha comma beta in R and consider the function.

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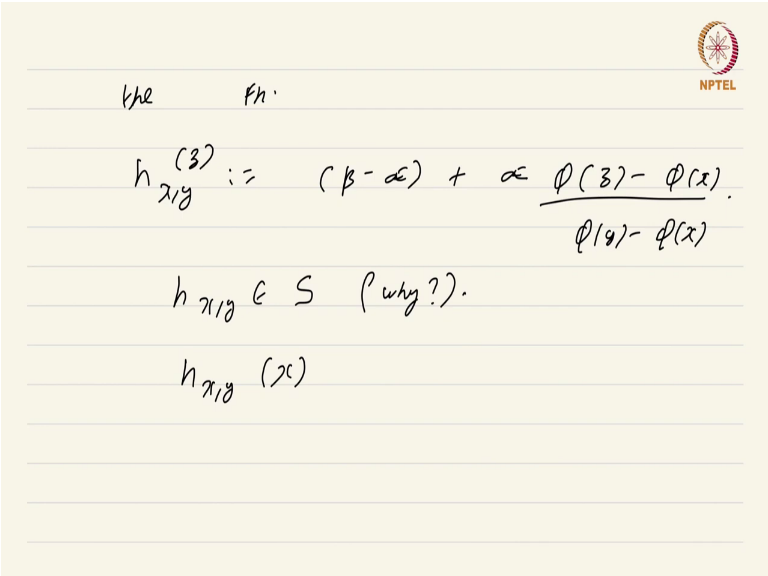

The function h of x comma y which is by definition equal to let us see whether I can get this right its beta minus alpha plus alpha times phi of z minus phi of x divided by phi of y minus phi of x ok. So, what is this trying to say? What am I trying to do? Well this is what I am trying to do. Let us go back to our picture we want to squeeze in this function g within this grid sort of thing centered on the function f.

So, what I am going to do is I am going to proceeds step by step I am going to fix points x comma y, I am going to fix points x comma y and my aim for the time being is to just construct one function in S that agrees with the given function F at the points x comma y.

So, the goal is I need to produce a function that is very close to the function F as a start what I am going to do is I am going to produce a function h x comma y which agrees with F at these two chosen points x and y that is the aim and the claim is that this function does the job for us this function does the job for us how does it do the job for us? Well, let us see what this functions value is first of all h of x, y is an element of S ok why? Please check that why is this an element of S.

Now, what is this functions what is the special feature of this function? At the point x; at the point x of course, this is a function of z. So, I must make that precise at the point x note that you get in the numerator phi of x minus phi of x.

So, the second term vanishes; second term vanishes and you are left with a beta minus alpha ok and at the point y; at the point y what happens is, let me just change this slightly; let me just change this slightly this is not what I want alpha plus beta minus alpha times this ok this is what I want.

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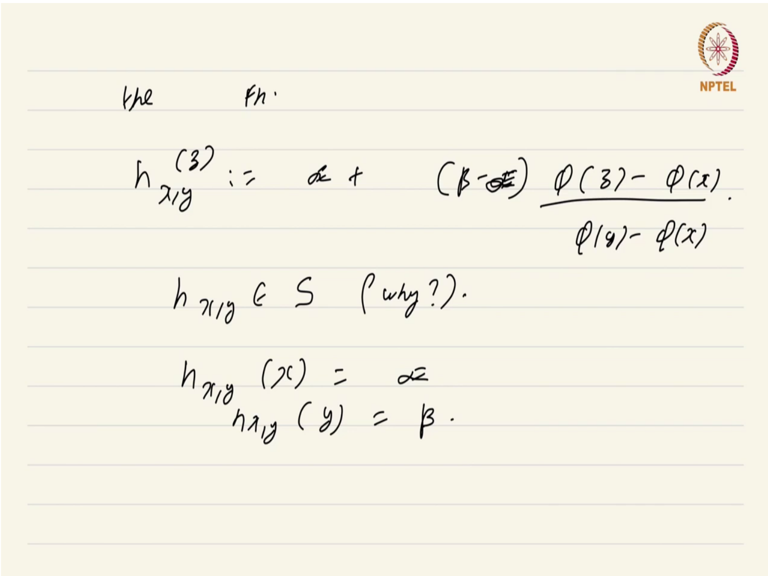

Now, let us see at x comma at the point x this thing vanishes. So, this whole thing vanishes. So, you will end up with alpha. So, this is just alpha and similarly h x, y at y is beta right because this quantity becomes phi of y minus phi of x by phi of y minus phi of x which is just one. So, this is alpha plus beta minus alpha which is just beta.

So, what have we achieved? What we have achieved is given two points given two points x comma y and values alpha comma beta I can find a function h x, y in S such that h x, y of x is equal to alpha and h x, y of y is beta. So, no matter what two points I choose in x and what two values alpha comma beta I choose on the real numbers, I can always find a function in S that at least agrees with I mean sort of interpolates h x, y x equal to alpha and h x, y y equal to beta ok.

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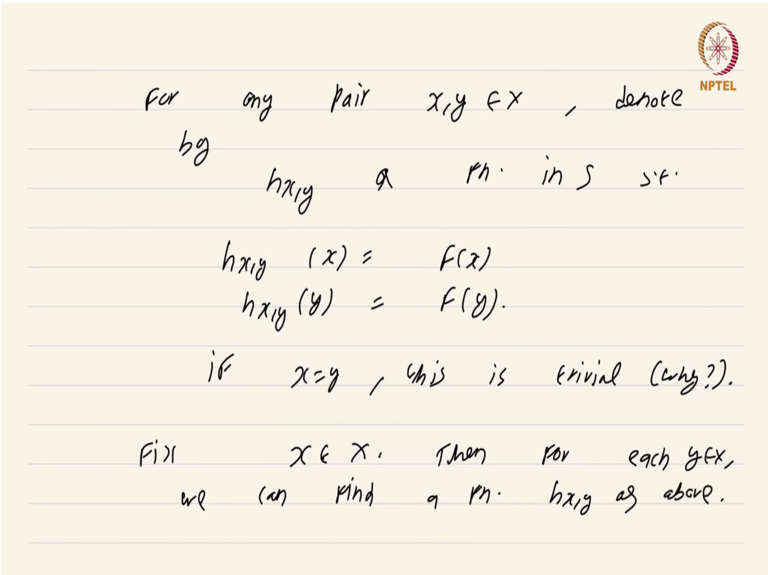

Now, for any pair x comma y in X denote by h x, y the function in S such that a function in S its it may not be unique a function in S such that h x, y of x is F of x and h x, y of y is F of y and the previous argument just shows that such a function h x, y exists for every pair.

Note if x equal to y this is trivial you do not need to use the previous argument ok why ok. So, what have I done? Given a pair of points x comma y I can find a function h x, y that agrees with F at the points x comma y. Now here is the part where I am going to use that minimum and maximum business fix x in X fix x in X now the goal is to at least satisfy this second set of this second inequality.

To produce a function g that is not too far above the function F it is lying below F of x plus epsilon ok how do I do that? Fix x in X then for each y in X we can find; we can find a function we can find a function h x, y as above as above ok.

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Let B_y be a ball centred at y

s.t.

$h_{x,y}(z) < F(z) + \varepsilon \ \forall z \in B_y$.

$h_{x,y} - F$

This can be done for each $y \in X$.

By compactness of X, we can

find $y_1, \dots, y_n \in X$ s.t.

$B_{y_1}, \dots, B_{y_n}$ covers X.

Now, let B y be the neighborhood or a ball be a ball centered at y such that h x, y of z is less than F of z plus epsilon for all z in this ball B y ok. What am I saying? Well we know that h x, y of y is equal to F of y what I am saying is choose a small ball call it B y centered at y such that h x, y of z is less than F of z plus epsilon for all z in B y. Why can I do this?

Well just consider the continuous function h x, y minus F this function is 0 at the point y therefore, by the standard epsilon whatever definition of continuity that you like you can find a ball such that h x, y minus F is less than epsilon in that ball which is same as saying h x, y z is less than F of z plus epsilon ok.

So, this is just a standard basic continuity argument because h and F agree at the point y, I can make h naught too far above F at least in this ball B y. This can be done; this can be done for each y in X right. For each y in X I have fixed X remember for each y in X there is a function h x, y corresponding to this point y there is also a ball B y such that the function h x, y on this ball B y satisfies this inequality ok.

So, this is all fairly straightforward, this is just using the basic properties of continuous functions. Consider h x by definition is minimum of wait a second I jumped by compactness by compactness by compactness of X, we can find; we can find y 1 to let us say y n in X such that B y 1 comma dot dot dot B y n cover X.

So, I am going to be very very brief here this is a standard compactness argument which should be second nature to you by now, we can find finitely many balls B y 1 to B y n that cover X ok.

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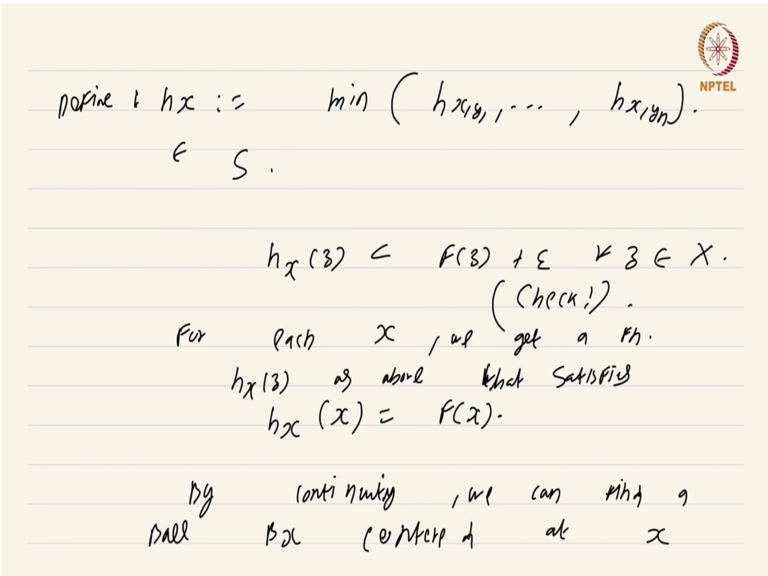

Look at the corresponding functions define h of x define h of x by definition equal to minimum of h x, y 1 comma dot dot dot h x, y n and by that induction and the additional assumption this h x is in an element is an element of S ok.

Now, what is the special feature of h x? Well it satisfies h x of z is less than F of z plus epsilon for all z in x ok check this is a trivial check by taking minimum we are more or less force this to happen in fact, because we want this to happen we have taken the minimum of the functions h x, y 1 to h x, y n ok.

So, as a recap we fixed x varied y and obtained these functions h x, y 1 to h x, y 1 y n we took the minimum we have got a function X. Now, we repeat the same trick again for each x we for each x we get a function; we get a function h x z as above that satisfies we have forgotten the point x. So, that is feeling a bit lonely satisfies h x x equal to F x right. Note that all of these functions h x, y 1 to h x, y n satisfy that h x, y 1 of x is x sorry is equal to F of x comma dot dot dot h x, y n of x is also F of x by taking the minimum we are not doing anything.

So, we still have this h x of x equal to F of x ok and for each x we can find a function such that h x of x equal to F of x and h x is not too far above F of z ok. Now we pull the same trick again define g to be max to be max ok again I am skipping a step I am skipping a step what we do is the following. By continuity by continuity we can find; we can find a ball B x such that F of x minus epsilon is less than sorry F of z minus epsilon is less than h F z h x of z for all z in B x ok.

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s.t. $F(z) - \varepsilon < h_x(z) \quad \forall z \in B_x$.

By compactness, we can find pts. $x_1, \dots, x_k$ s.t. $B_{x_1}, \dots, B_{x_k}$ cover X.

Set $g(z) := \max(h_{x_1}(z), \dots, h_{x_k}(z))$.

By construction $F(z) - \varepsilon < g(z) < F(z) + \varepsilon \quad \forall z \in X$.

So, we can find a ball B x centered at x; centered at x such that F of z minus epsilon is less than h x of z for all z in b x. This is the same continuity argument again by using the fact that this function h x and F are both continuous this is rather straightforward to see ok. Now by compactness we can find; we can find points x 1 to let us say x k such that B x 1 comma dot dot dot B x k cover X.

So, for each x we can find a ball B x such that this happens the collection of all such balls cover x there is a finite sub cover choose x 1 to x k such that B x 1 to B x k cover X. Now we are almost done set g of z by definition to be max of h x 1 z comma dot dot dot h x n h x k z ok then by construction F of x sorry F of z minus epsilon is less than g of z is less than F of z plus epsilon for all z in X ok.

So, this completes the proof please ponder over the proof for some time to understand what is happening its just rather you do the most straightforward and obvious thing and everything just works out. Now we have to justify getting rid of the additional assumption ok we have to get rid of the additional assumption.

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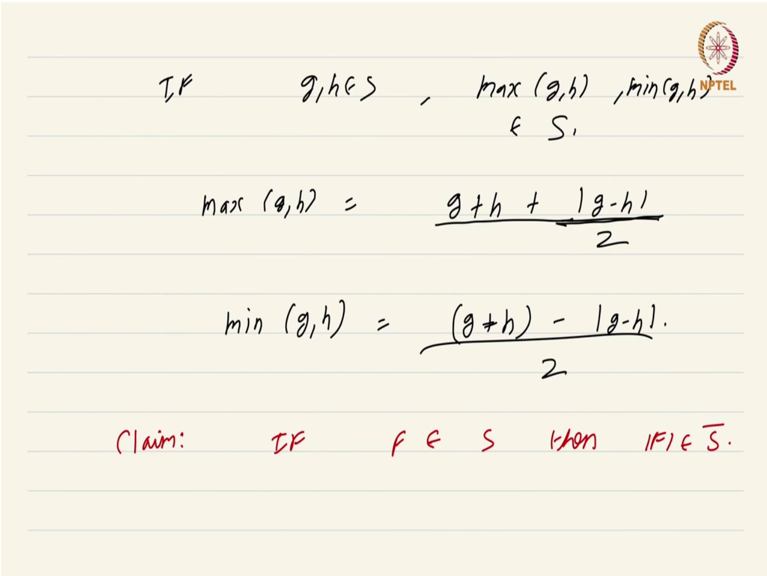

What did the additional assumption say? It says that if F comma g or rather g comma h are in S then we assume that max of g comma h and min of g comma h are both in S ok. This is not quite always true, but we can get rid of this quite easily what we do is the following. First of all observe that max of g comma h is just g plus h plus mod g minus h divided by 2 ok I think this whole thing is divided by 2 and minimum of g comma h is g plus h; g plus h minus mod g minus h the whole thing divided by 2 ok.

Let us see is this correct max g h is g plus h plus mod g yes it need it is correct. So, please check that these formulas for the maximum and minimum of g and h are correct I think I have done this once before in real analysis one if not its a straightforward check ok.

So, we have explicit formulas for g plus h sorry max of g h and minimum of g h, now observe that because we are in an algebra where S is actually a sub algebra if you could show that modulus of g minus h is in S you are actually done.

So, what we will now prove is not that will not be true for any sub algebra that satisfies the other properties, what I am going to show is I am going to prove claim if F is in S, then mod F is in S closure I am going to show this instead. Then I am going to leave it to you to see that this claim will immediately get rid of the additional assumption that additional assumption is not really going to make that much of a difficulty for us ok.

So, I am not going to give the full proof intentionally, I am going to leave a bit of thought to you I am going to show that if F is in S then mod F is there in S closure ok.

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$F \in C(X)$, therefore it is bounded.

$C > 0$

$$-C \leq F(x) \leq C \quad \forall x \in X.$$

Observe that $|\cdot|$ on $[-C, C]$ is uniformly approximated by some polynomials. P_n.

$$P_n \to |\cdot| \text{ on } [-C, C].$$

$$P_n(F) \longrightarrow |F| \text{ on } X \text{ (Show!)}.$$

This shows $|F| \in \bar{S}$. Use this to finish the proof.

How do you show this? Well F is first of all a continuous function therefore, therefore, it has it is bounded because x is compact therefore, it is bounded it is bounded. So, we have minus C less than or equal to F of x is less than or equal to plus C, C is greater than 0 for all x in X ok.

Now what you do is observe that the absolute value function on the closed interval minus C C is uniformly approximated by some polynomials let us say P n; let us say P n. So, we have P n converges to the absolute value function on minus C, C ok.

Now, because S is an algebra P n with the variable x substituted by the function F ok. I will leave it you to figure out what this means, this is if you have taken any course on linear algebra this should be obvious to you what it means. P n with the variable X substituted by the function F converges to the absolute value of F on x show this is trivial show this ok.

So, this shows mod F is there in the closure is in S closure ok use this to finish the proof to finish the proof.

So, I am intentionally leaving a number of simple checks for you so, that your understanding is thorough. This will be the common theme for the rest of the course a number of such things which will not take more than 15 20 minutes for you to do on your own I am going to leave it to you so, that we can concentrate on the more difficult material.

So, this concludes the proof of the Stone-Weierstrass theorem, there are some analogues of this and some exercises in the notes please go through them and solve them so, that you have a solid understanding. This is a course on real analysis and you have just watched the video on the Stone-Weierstrass theorem.