Real Analysis II
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Lecture - 8.2
Upper and Lower Limits

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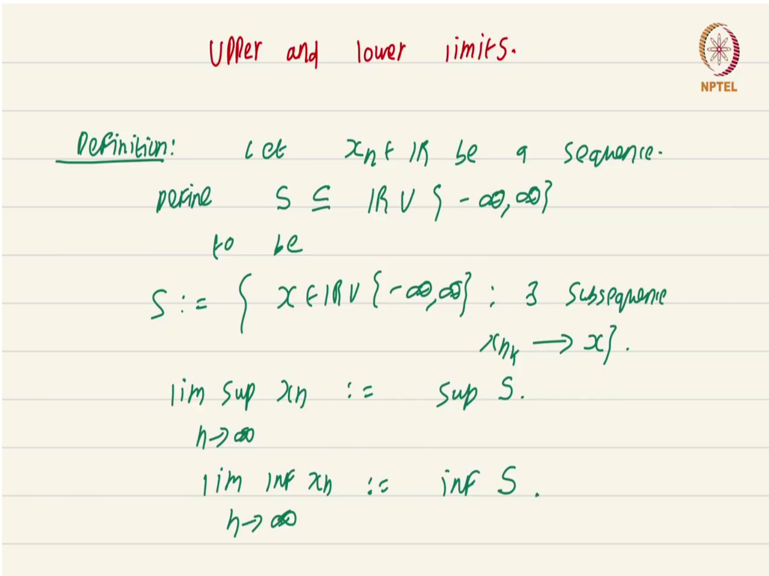

In this video we are going to study Upper and Lower limits which are more popularly known as lim sup and lim inf. They are very useful in the study of sequences that are actually not convergent. So, the definition is quite straightforward, definition let x n in R be a sequence define defined S which is actually a subset of R union minus infinity infinity the extended real numbers with the points minus infinity and infinity added.

I remind you again that these are not real points they are just a notation, to be set of all x in R union minus infinity infinity such that there is subsequence x n k that converges to x or rather

I mean it could converge to x if it is a real number or it diverges. If you want the point infinity to bear in be in this set S, there should be a subsequence that diverges to plus infinity. Similarly, if you want minus infinity to be there in this set you need a sequence x n sub sequence x n k that diverges to minus infinity.

Now, the lim sup of x n; n going to infinity is just by definition supremum of the set S, if this set S is unbounded or if this set S contains the point plus infinity the supremum is taken to be plus infinity. Similarly, lim inf x n; n going to infinity is just the infimum of the set S with the same remarks, if the set S is either not bounded below or it contains the point minus infinity then the lim inf is just minus infinity.

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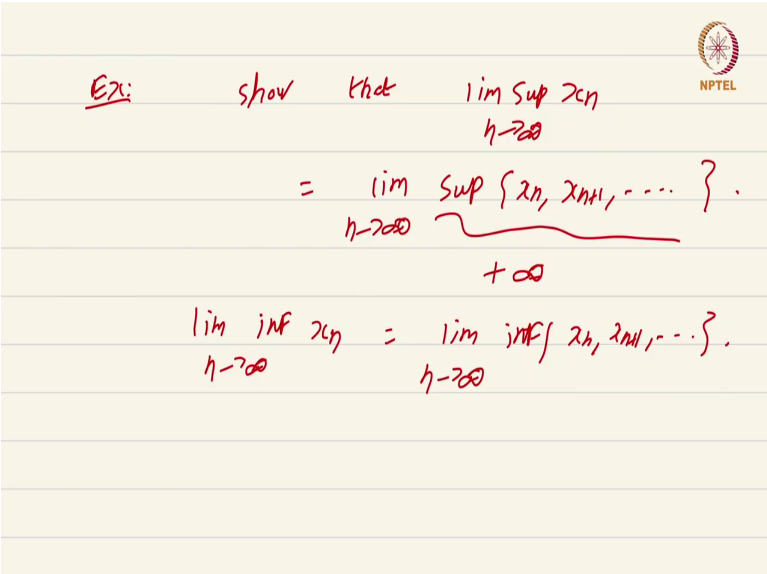

So, immediately you can characterize this lim sup and lim inf in several ways, I am going to leave one exercise for you. Show that lim sup n going to infinity x n is just equal to limit n going to infinity supremum or rather limit n go yeah this is correct, limit n going to infinity supremum of x n, x n plus 1 comma dot dot dot. You just start taking the supremum's of the tails and then take limit n going to infinity.

So, there is one just this one remark that needs to be made, this could be plus infinity ok. So, you will have to be a bit careful when you analyze and prove this exercise, you will be taking limits where some of the terms could be infinity. So, I want you to handle this with a bit of delicateness it is not that difficult, but let me just tell you that this could be plus infinity. And not only that if it is going to be plus infinity for some n it is going to be plus infinity throughout, that will give you the key.

So, it is just going to be a constant plus infinity, it cannot happen that supremum x n x n plus 1 is plus infinity, but when you remove finitely many terms suddenly the supremum becomes finite that is not going to happen ok. Similarly, lim inf n going to infinity of x n is just limit n going to infinity of infimum of x n, x n plus 1 and so on ok.

So, these are the basic properties of, I mean this is one of the basic properties of lim inf and lim sup, you can characterize it in terms of both sub sequential limits as well as this definition. Now, about characteristic properties there is a very good characteristic property of lim sup and lim inf that you must know, that you must know and this formulation is what is used in proofs most often.

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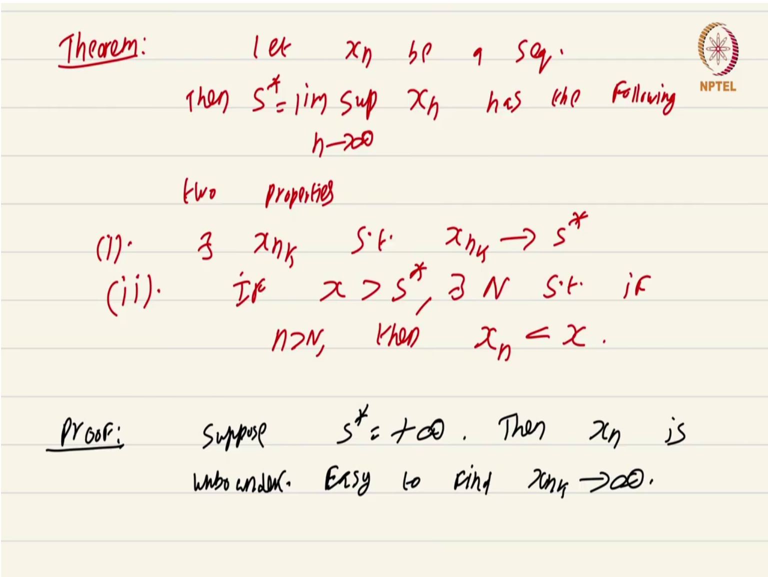

Theorem: let x n be a sequence; let x n be a sequence then lim sup n going to infinity of x n has the following 2 properties; the following 2 properties. Property number 1, there exist a subsequence x n k such that x n k if you do not mind let me just call this lim sup S upper star this is a common notation x n k converging to S upper star ok.

Property number 2 if x is greater than S upper star, if you choose a real number that is greater than S upper star there exist n, such that if n is greater than n then x n is less than x ok.

So, in some sense the lim sup is not an upper bound for the sequence it is an essential upper bound, after a particular point you cannot become greater than, I mean you cannot become greater than any number greater than the lim sup ok that is a mouthful. But it is a very very intuitive property. The sequence will not be too far greater than lim sup provided you go far enough in the sequence.

So, let us prove this characterization. Proof: suppose S upper star is plus infinity ok. Then the only thing the only way by which this can happen is then x n is unbounded, it is an unbounded sequence, it is easy to find x n k diverging to plus infinity and i am going to leave it to you ok.

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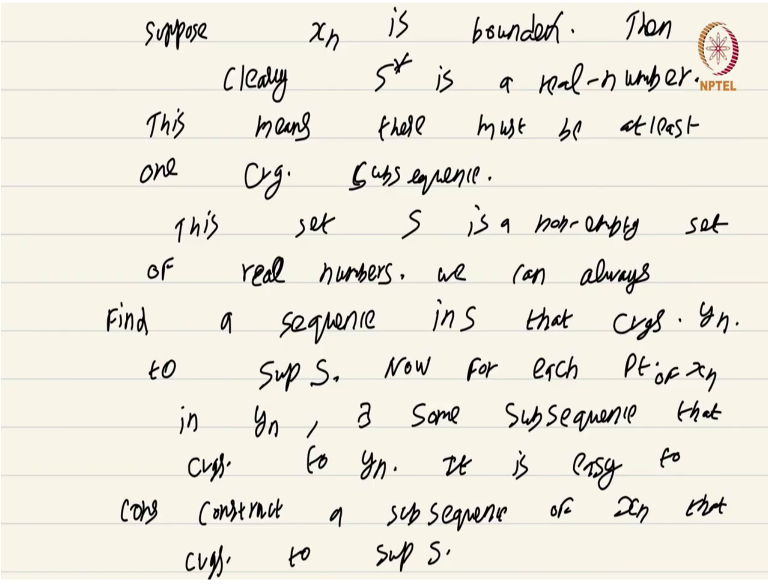

Now suppose x n is bounded x n is a bounded sequence then clearly S star is a real number obviously ok. Now, this means there must be at least; there must be at least 1 convergent subsequence this is just the Bolzano Weierstrass theorem convergence subsequence, any bounded sequence must have a convergence subsequence ok.

So, this set S is non empty; is a non empty set of real numbers; set of real numbers ok and we already know that given any non empty set of real numbers we can always find, we can always find a point a sequence.

Let us say not just a point we can always find, we can always find a sequence; we can always find a sequence in S that converges to supremum of S oh. I must mention this S, set S is the same that we saw in the definition of lim sup I have defined a set S I am going to reuse the notation ok.

Now, what I am trying to say is that given this set S is a nonempty and bounded set, we can always find a sequence in S that converges to supremum of S, this is always true. Now, for each point in the sequence, so let us just call this sequence y n ok. Let us just call this sequence y n, for each point in y n there is some subsequence that converges of x n that converges to y n ok. Now, it is easy it is easy the very similar arguments we have done several times before it is easy to construct a subsequence of x n k of x n that converges to supremum of S.

This is something that we have done several times earlier, some very similar arguments. So, I am going to leave the rest to you ok. So, there is a third possibility that we should consider.

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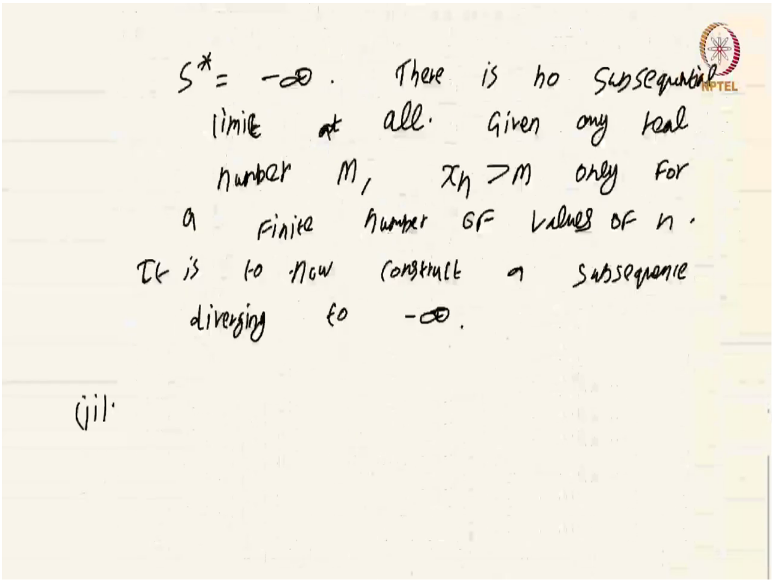

What if this set of subsequential ah limits that is a empty set ok not an empty set, it consist of only minus infinity that is possible, suppose S star is minus infinity what if the lim sup itself is minus infinity ok.

This just means there is no subsequential limit; there is no subsequential limit whatsoever limit at all ok. That just means for given any real number, given any real number m given any real number m x n is greater than m only for a finite number of values number of values of n. Why is this the case? Well think about it this way, if it is not true that x n is greater than m only for a finitely many values of n then that just means that there is a subsequence of x n that is strictly greater than n.

Now, there are two possibilities, this subsequence could be bounded in which case we can apply the Bolzano Weierstrass theorem to get a further subsequence that converges contradicting the fact that there is no subsequential limit at all. On the other hand if this x n is unbounded then you would get a subsequence that diverges to plus infinity, again contradicting the fact that there is no subsequential limit whatsoever ok.

So, what this shows is that only finitely many values of n are for only finitely many values of n x n is greater than M, it is easy to now construct, it is easy to now construct a subsequence diverging to minus infinity; diverging to minus infinity ok. So, this will provide for you the subsequence that diverges to minus infinity.

Now, for on to part the second part, let us just recall the second part asks us to show that if a number is greater than S star, then there exist n such that if n is greater than n then x n is less than x ok. Now, again we can use a similar logic to what we have seen so far to the following.

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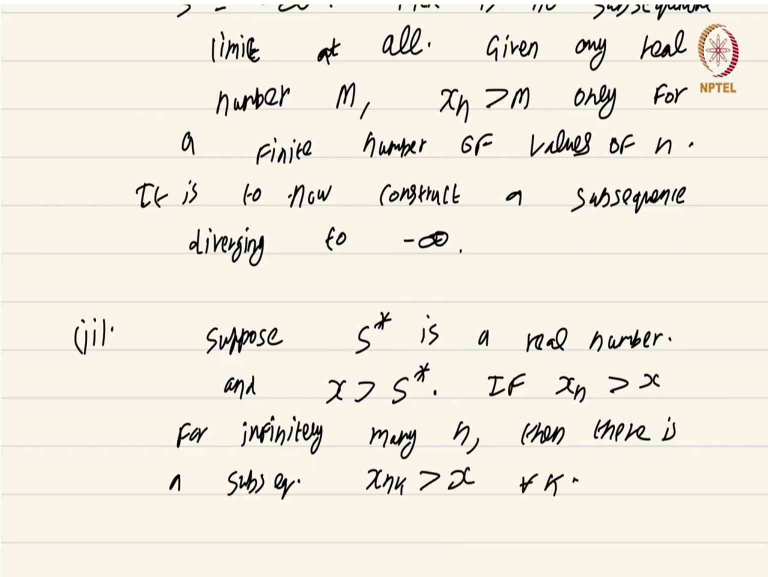

Suppose this S star is a real number, it is a real number, of course if S star is plus infinity there is really nothing to show ok. So, suppose S star is a real number and x is greater than S star ok, then if x n is greater than x for infinitely many values infinitely many n then there is a subsequence, again same logic there is a subsequence x n k which is strictly greater than x for all k right.

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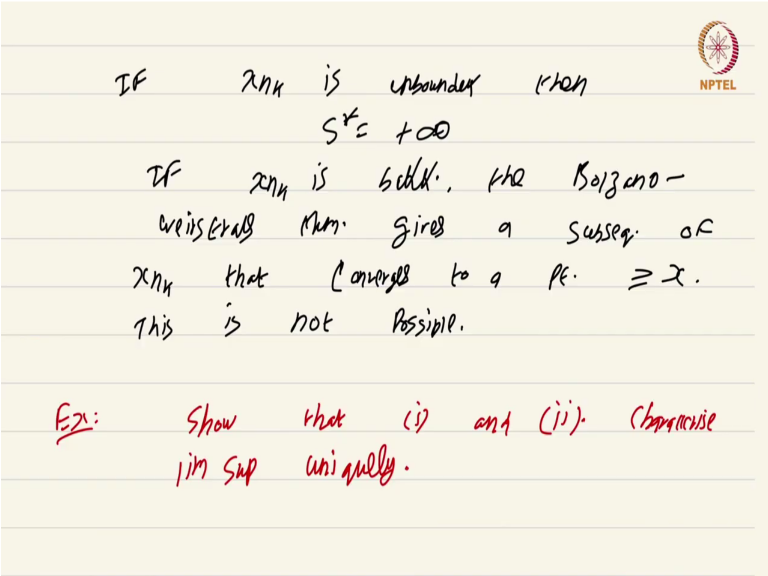

Now, there are two possibilities, if x n k is unbounded, then in fact S star is plus infinity which is nonsense. On the other hand if x n k is bounded, the Bolzano Weierstrass theorem; the Bolzano Weierstrass theorem Weierstrass theorem guarantees or let me just say gives a subsequence of x n k that converges to a point greater than or equal to x, which is again this is not possible. Because x is the supremum of all subsequential limits ok.

Now, the case where x is just, S star is just minus infinity is very easy, if S star is minus infinity it is rather trivial to show this ok. Now, this concludes actually concludes the proof of property 1 and 2 I just want to make a remark or rather I will give an exercise show that 1 and 2 characterize lim sup uniquely. That means, if there is a number that satisfies 1 and 2 then that has got to be the lim sup ok.

So, I am going to leave further exercises in the lecture notes where you extend the root test and the ratio test, make it slightly stronger incorporating our knowledge of lim sup and lim inf. And I am also going to leave the Cartan Hadamard theorem for the radius of convergence of a power series which is just a straightforward application of the root test to you. You will be able to find out compute a formula for the radius of convergence of a power series.

There are some more examples also of these notions in the lecture notes. This is a course on real analysis and you have just watched the video on upper and lower limits.