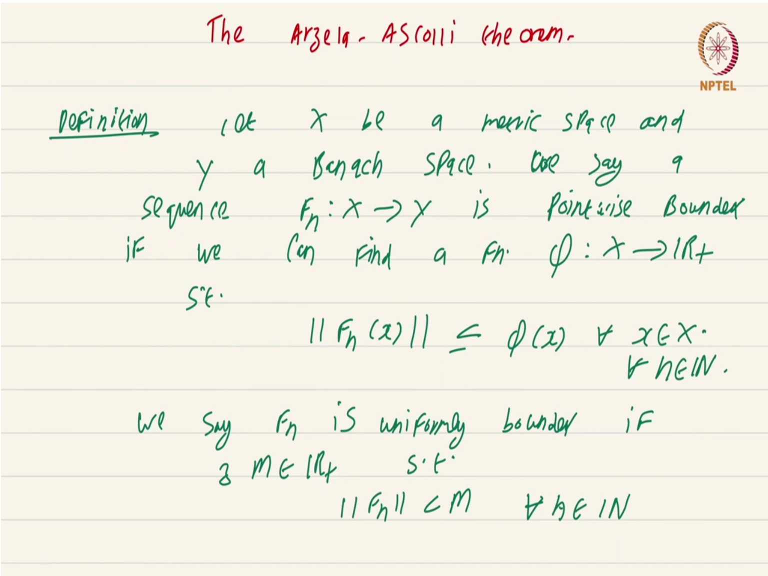
**Real Analysis II**
**Prof. Jaikrishnan J**
**Department of Mathematics**
**Indian Institute of Technology, Palakkad**

**Lecture - 8.1**
**The Arzela--Ascolli Theorem**

(Refer Slide Time: 00:23)

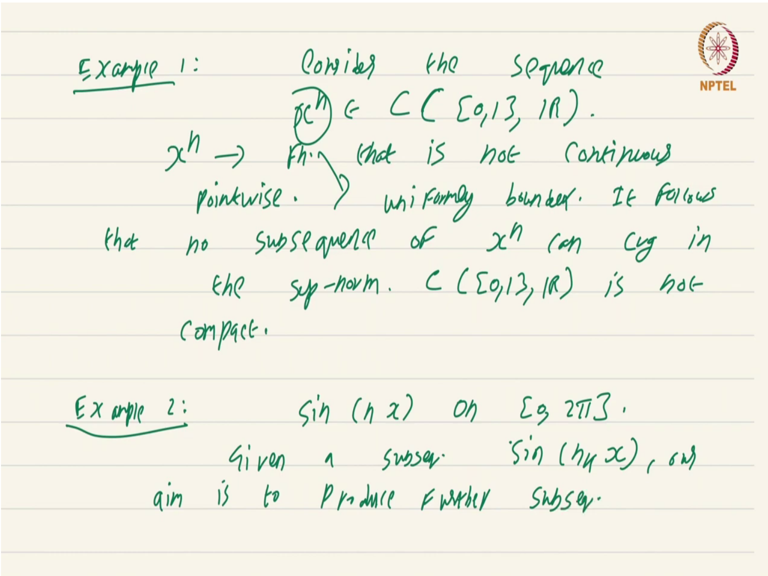



The Bolzano Weierstrass Theorem guarantees, that any bounded sequence of real numbers always has a convergent subsequence. The question arises is it true for a sequence of functions? Suppose, you have a sequence of bounded functions, does it necessarily mean that there is a convergent subsequence and what is the meaning of convergent here?

Let us first clarify that. Definition, let X be a metric space and Y a banach space and y a banach space. Consider a sequence or we say a sequence, we say a sequence, F n from X to Y is point wise bounded; is point wise bounded. If we can find a function phi from X to R plus such that norm F n x is less than or equal to phi of x for all x in X. We say F n is uniformly bounded, if there exist M in R plus such that norm F n is less than M for all n in N ok.

This first inequality is also for all n in N. So, point wise bounded just means that at every point the sequence F n x you can find a bound for it. Uniformly bounded just means that the norm of F ns itself are uniformly bounded by the single number capital N. Let us see a couple of examples to see what is going on.

(Refer Slide Time: 02:55)




Example 1consider the sequence x power n which are elements I am going to treat them as elements of C close 0 1 R ok. Now, we already know that close 0 1 I mean x power n converges to a function, to a function, that is not continuous point wise ok.

So, these are a sequence of functions that are uniformly bounded, these are a sequence of functions that are uniformly bounded right. And, they are uniformly bounded by just the number 1. Now, because if you take a subsequence of x power n, that will also have to converge to this same discontinuous function, it follows that no subsequence, it follows that no subsequence of x n, x power n, can converge in the sup norm right.

Because, convergence in the sup norm is same as uniform convergence, and uniform convergence would guarantee that the limit function is continuous. Since, the limit function cannot possibly be continuous, the functions x power n cannot converge uniformly no subsequence can also have this property. So, concisely capture this because compactness and sequential compactness are the same, we can conclude that C 0 1, R is not compact right.

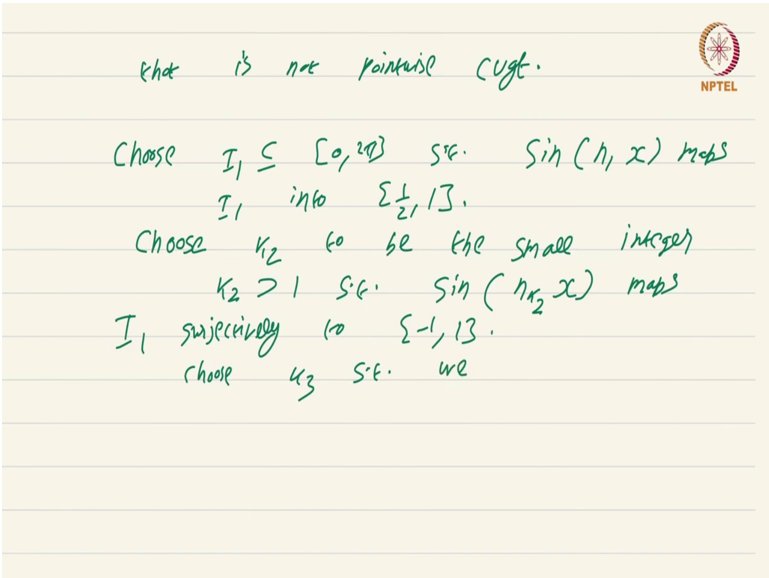
So, here is an example where, if you have uniform boundedness you do not get a convergent subsequence and compactness also fails. Now, a deeper question might be ok here we did have point wise convergence at least is that at least guaranteed all the time? No.

So, that is example 2, which is a lot more involved. Let me just make a remark that later in this course, you will study the Lebesgue dominated convergence theorem, after which this will be very very easy to prove. But, without some sort of convergence theorem from measure theory you will have to appeal to some sort of clever argument, that is what we are going to do now. Consider the sequence sin n x defined on close 0 2 pi ok.

What I am going to do is the following. I am going to produce, I am going to produce for any given subsequence, for any given subsequence, I am going to find a way to produce a further subsequence such that at some point it will not be pointwise convergent.

So, let me repeat that again so, given a subsequence, given a subsequence, sin n k x, I am going to produce a further subsequence. So, aim is to produce, that our aim is to produce, aim is to produce further subsequence that is not pointwise convergent ok, that is not pointwise convergent ok.

(Refer Slide Time: 06:35)



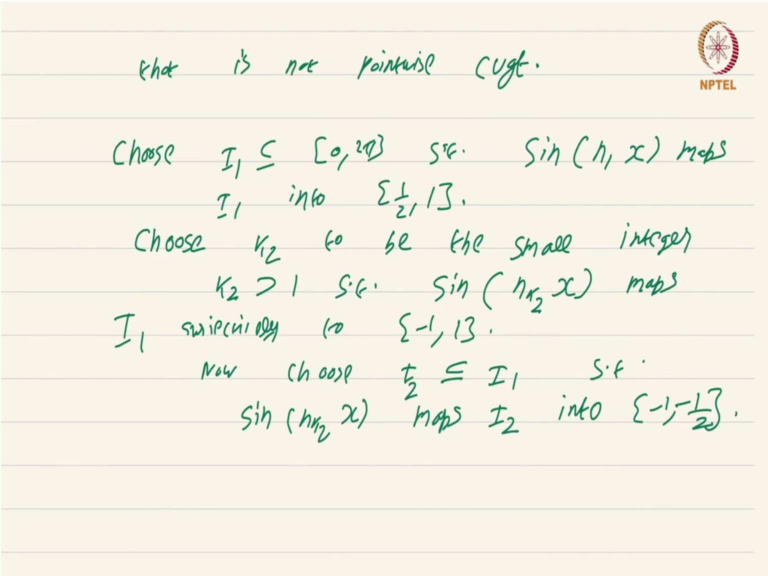

So, what we are going to do is the following. We are given this sin n k x, note that this n ks being natural numbers that increase as k increases, these n k sort of expand any given interval and makes it larger and larger.

So, what we are going to do is the following? Choose I 1 subset of 0 2 pi such that, sin n 1 x maps I 1 into half 1 ok. Choose some small sub interval of 0 to 2 pi such that sin n 1 x maps I 1 into half 1 this is very easily doable please think about, why this is possible.

Now, what we are going to do is choose, n 2 to be or rather choose k 2 to be the smallest integer, the smallest integer k 2 greater than 1, such that sin n k 2 of x maps surjectively, surjectively to close minus 1 ok.

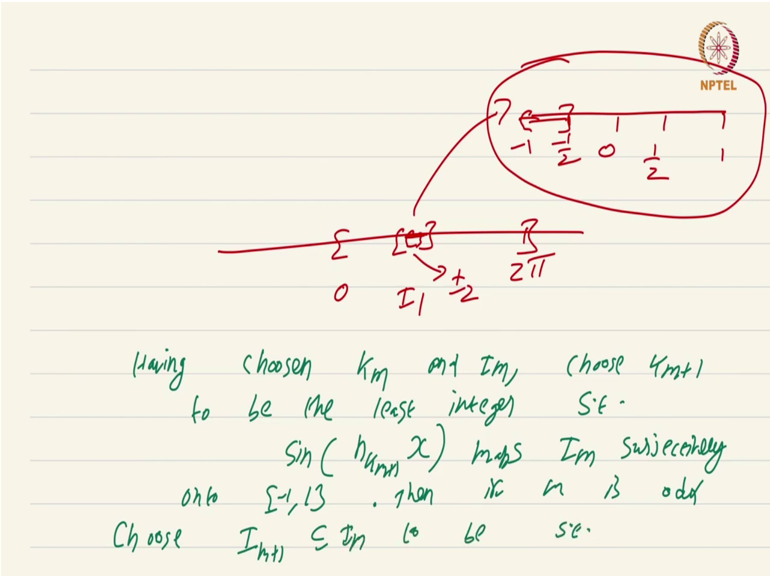
What we are doing is maps I 1, I should mention that, otherwise it makes no sense, we have already chosen the small interval I 1, such that sin n 1 x maps I 1 to half 1. Now, what we are doing is we are choosing k 2 so, large such that sin n k 2 x expands, this small interval I 1 to be a large enough interval that contains an interval of length 2 pi. So, that sin n k 2 x will map I 1 surjectively to minus 1 1, this is certainly possible ok.

(Refer Slide Time: 09:25)




Now, what you do is choose the next integer choose k 3, such thatwe can find or rather what you do is. Now, chooseI 2 subset of I 1, subset of I 1. Such thatsin n k 2 of x maps, maps I 2 into minus 1 half, minus 1, minus half ok. So, this might seem a bit confusing what is happening. Let us draw a picture to make sure that we understand really what is going on.

(Refer Slide Time: 10:07)



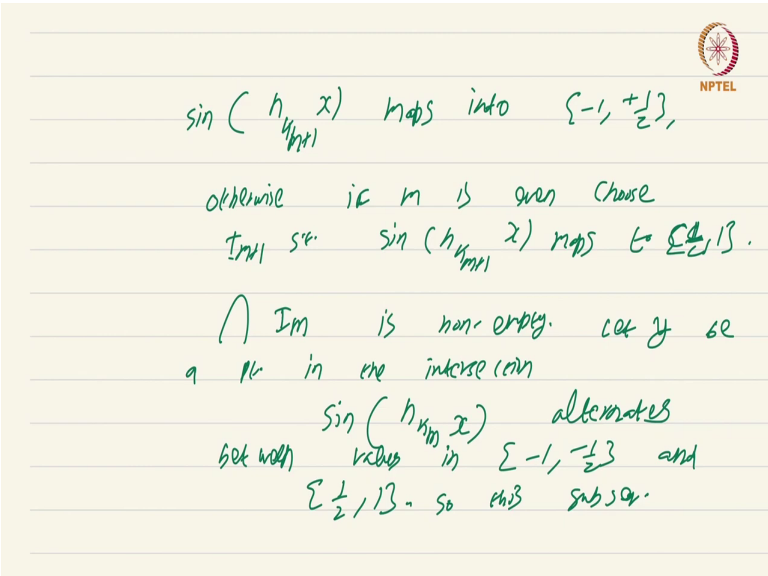

This is what we are doing; we have this 0 to 2 pi interval ok. What we are doing is we have first choosing a small sub interval I 1, such that this I 1 gets mapped to half 1, it gets mapped to half 1. Let me just make this bigger, because I want to include the negative points also. So, let me just make this a bit bigger.

So, somewhere here half 1, 0 is here ok. Now, what you do is you choose k 2 to be so, large that sin n k 2 of x again covers this whole minus 1, 1 it surjectively maps to minus 1 1. Now, choose a subset I 2, a small subset I 2, small subset I 2, such that sin n k 2 of I 2 maps to this portion, minus 1, minus half, this portion ok.

Now, you can guess what is going to happen, we are going to consecutively choose smaller and smaller intervals and larger and larger case. So, that this alternating between half 1 and minus 1 half keeps happening.

So, to be precise what you do is having chosen K m, having chosen K m, find having chosen K m and I m choose K m plus 1, 1 to be. The least integer, least integer, least integer such that sin n K m plus 1 x maps, maps I m surjectively, surjectively to on to minus 1 1 ok. Just boost up the index. So, that I m gets maps surjectively on to minus 1 1. Then, if m is odd, if m is odd, m is odd, choose I m plus 1 subset of I m to be such that, to be such that sin n K m plus 1 of x maps into minus 1 half.

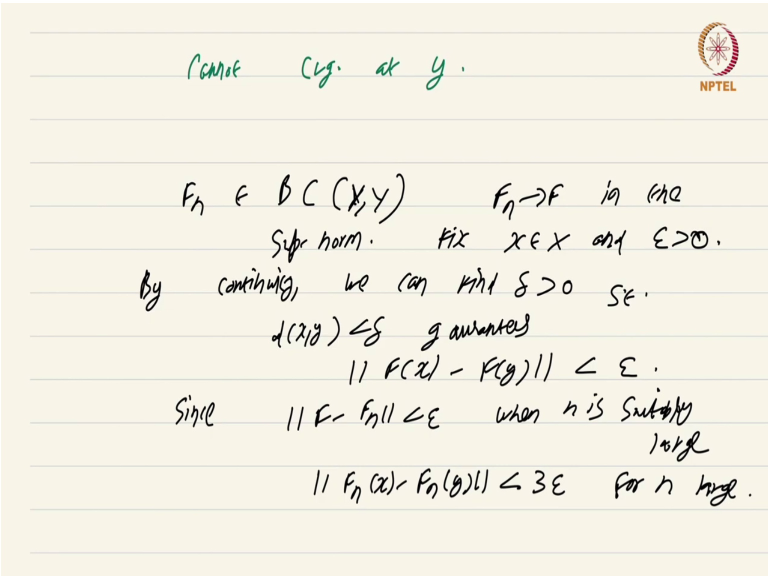
(Refer Slide Time: 12:57)



Otherwise, if m is even choose I m plus 1 such that, sin n k plus n k m plus 1 x maps to half 1 ok. Depending on whether it is odd or even, the index m you just keep alternating between half 1 and minus 1 minus half.

Now, you will understand in a moment why we are doing this complicated business. Why the nested intervals theorem this intersection of I m is non empty; is non empty ok. Let y be a point in the intersection, point in the intersection. Now, it is obvious that sin n k m of x alternates between the values in minus 1 minus half and half 1ok. So, this subsequence cannot converge, this subsequence cannot converge at y.

(Refer Slide Time: 14:45)




So, what we have shown is no matter what subsequence you choose, we can find a further subsequence, that does not converge pointwise at this given point y. Note these sequences sin n x are all uniformly bounded by 1. So, uniform boundedness is not enough to guarantee, that a sequence of functions will have a convergent subsequence.

So, we need something deeper than uniform boundedness and what that is called is equi continuity. To illustrate what equicontinuity is suppose you have F n in this B C X Y bounded and continuous functions from a metric space to a banach space and such that F n converges to F in the sup norm. Suppose, we are in the situation, this the most general situation that we are interested in, fix x in x and epsilon greater than 0, fix x in x and epsilon greater than 0 ok.

By continuity we can find, we can find, delta greater than 0, such that d x y is less than delta guarantees norm F x minus F y is less than epsilon, this is just the epsilon delta definition written out in all it is glory.

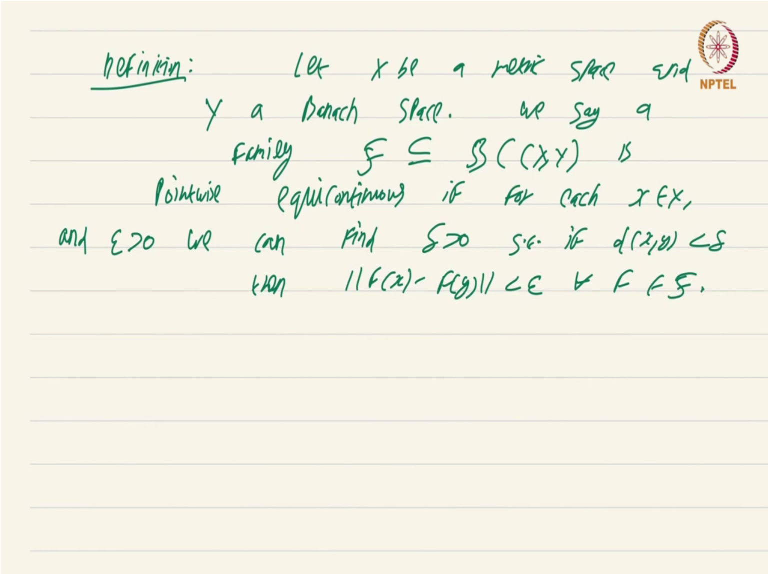
Now, since F minus F n norm is less than epsilon when n is suitably large, because we are taking F n converging to F in the sup norm when n is suitably large, when n is suitably large, what this shows is, we can use this epsilon by 3 trick that we have been using at least a half a dozen times in this course so, far to show that norm F n x minus F n y is also less than 3 epsilon, for n large ok.

So, what we have essentially done is we have fixed a point x and an epsilon greater than 0. Using the fact that F is continuous, we have found a delta that works in the epsilon delta definition and what we are saying is that this same delta works for F ns, when n is suitably large ok.

In some sense all these F ns are uniformly continue, not just uniformly continuous, they are sort of continuous in a way that is independent of n. In other words they are uniformly continuous, sorry equi continuous that is the technical term they are equi continuous ok.

Now, I am going to leave a definition of equi continuity and leave it to you to check that any finite collection of functions will automatically be equi continuous in this definition and therefore, you can show, that whenever you have F ns converging to F in the sup norm they have to be equi continuous. I am going to leave that to you let us move on to the definition to the definition of equi continuity.

(Refer Slide Time: 18:11)




Definition; again the setup is X is a metric space, let X be a metric space and Y a banach space and Y a banach space. We say a family F subset of B C X Y bounded and continuous functions. Is pointwise equi continuous, is pointwise equi continuous, if for each x in X, we can find and epsilon greater than 0; for each x in x and epsilon greater than 0, we can find delta greater than 0, such that if d x, y is less than delta, then norm F x minus F y is less than epsilon for all functions coming from this family.

So, the choice of delta corresponding to epsilon is independent of the function coming from the set F, then we say the set family F is point wise equi continuous. If, you could choose this delta independent of the point also, then you say the family entire family is equi continuous, without any further adjective.

So, the main theorem we are interested in is the Arzela Ascolli theorem, there are many many many scenarios where this theorem is applicable and there are many slightly varying versions of this. I am going to present one version that is most suitable for applications, that is when the domain X is a compact metric space and Y is a Banach space ok.

So, in particular when you have a compact metric space and a banach space, these functions F will all be bounded at least; by compactness I hope you have solved that exercise that was given to you earlier ok. So, without further ado let me state and prove the Arzela Ascolli theorem. There are several proofs as well most of them involve a diagonal argument I am going to present an argument that does not involve any diagonalization procedure ok.

(Refer Slide Time: 20:39)

<u>Arzela-Ascoli theorem</u> Let $X$ be a compact metric space and $Y$ a Banach space. A Family $\mathcal{F} \subseteq \mathcal{C}(X,Y)$ is relatively compact iff it is pointwise equi continuous and each orbit

$$\mathcal{F}(x) := \{ F(x) : F \in \mathcal{F} \}.$$

is relatively compact in $Y$.

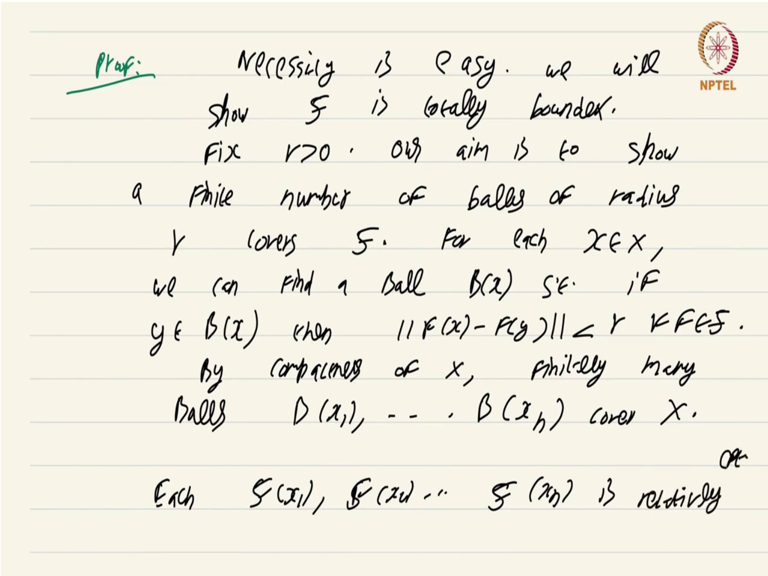
So, here is the statement of Arzela Ascolli theorem, Arzela Ascolli theorem. So, let X be a compact metric space, let X be a compact metric space, and Y a banach space and Y a banach space. A family, family F subset of B C X, Y is relatively compact this B is sort of redundant in our case in any case simply, because we are starting from a compact ah space.

So, is relatively compact, remember this just means that the closure is compact, if and only if it is pointwise, pointwise equi continuous, pointwise equi continuous. And, each orbit, I will get into a moment what this orbit business is. Each orbit fx which is just defined to be what x maps to under the various elements of F, that is just F of x such that F is in F.

So, you should visualize it as a set of points where this family F takes the given 0.2 ok. And, each let X be a compact metric space and Y be a banach space, a family F subset of B X Y is relatively compact, if and only if it is pointwise equi continuous, and each orbit F of X is relatively compact in Y ok.

So, checking whether a family of functions that are continuous on a compact metric space, mapping into a banach space, you have to check equi continuity and you have to check that each orbit is relatively compact.

(Refer Slide Time: 22:49)




So, let us go on to the proof, there is a slight trick in the proof. So, proof so, necessity is left to the reader, necessity is easy. In fact, we have already seen aspects of this necessity, before we motivated the definition of equi continuity. Now, we are going to use one of the criteria for compactness that we have done, that is total boundedness, we are going to use total boundedness.

We will first show, we will show, we will show, that this script F is totally bounded. That will actually finish the whole thing by the general Heine Borel Theorem and the fact that B C X Y is complete ok.

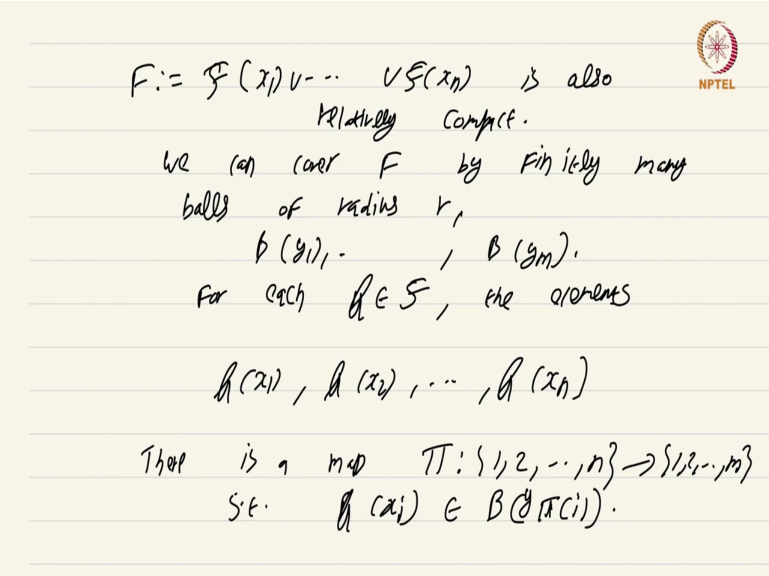
Now, to show equi bounded total boundedness, we will fix r greater than 0 ok. Now, we have to show that a finite number of balls of radius r covers the set F. So, our aim is to show aim is to show, a finite number of balls of radius r of radius r covers f ok.

So, what we do is the following. For each x, for each x in X, we can find a ball, we can find a ball, we can find a ball, we can find a ball B of x. Such that, if y is an element of B of x; then norm F x minus F y is less than r for all F in f ok.

This is just equi continuity point wise equi continuity what I am doing is first we are going to use equi continuity. To choose a ball B of x for each point, such that if y is there in this B of x, norm F, x minus F y is less than r for all F in f. Remember the goal is to cover the entire set F by finitely many balls in the space B C X Y, remember that ok. Now, by compactness, by compactness of X, finitely many balls, finitely many balls B x 1 dot dot dot B x n cover x ok.

Now, we are going to focus on the orbits of these points x 1 to x n. And, we are going to exploit the fact that these orbits are all relatively compact to find out finitely many balls, in the space B C X Y that cover F ok. Now, each F x 1, F x 2 dot dot dot, F x n is relatively compact, that is the hypothesis relatively compact ok.

(Refer Slide Time: 26:17)




Now; that means, their union F of x 1 union dot dot dot F of x n is also relatively compact, is also relatively compact. A finite union of relatively compact sets is relatively compact, this is very easy to see ok.

Now, we can cover, we can cover call this finite union capital F, we can cover F by finitely many balls, by finitely many balls, many balls of radius r. Why is that? Because, by the Heine Borel theorem or whatever, this F is being compact, I mean F being relatively compact F closure is compact and therefore, you can cover by there is no Heine Borel theorem sorry about that. Just the fact that F is relatively compact, will mean F closure is compact, you can cover F closure by finitely many balls of radius r therefore, you can cover F as well ok.

So, we can cover F by finitely many balls of radius r. Let us say we let us call these balls B y 1 dot dot dot B y m ok. Note that, this y 1 to y m might have nothing to do to nothing to do with x 1 to x m ok. Now, we are going to relate y 1 to y m to x 1 to x m that is the key of the proof.

What is the meaning of b y 1 to y m covers F? It just means that for each F F in script f, for each F in script f the elements the elements F x 1, F x 2. So, let me just for consistency let me just write this F in the same way so, that I do not confuse you. So, F x 1, F x 2, dot dot dot F x n ok.

This you pick an element from script F we know that the entire orbit, which we have called capital F is covered by b y 1 to b y m. That means, each one of these elements F x 1, F x 2, F x m they will have to belong to one of these each one of them right. They may not all belong to the same for obvious reasons, but they will have to belong to one of them right.

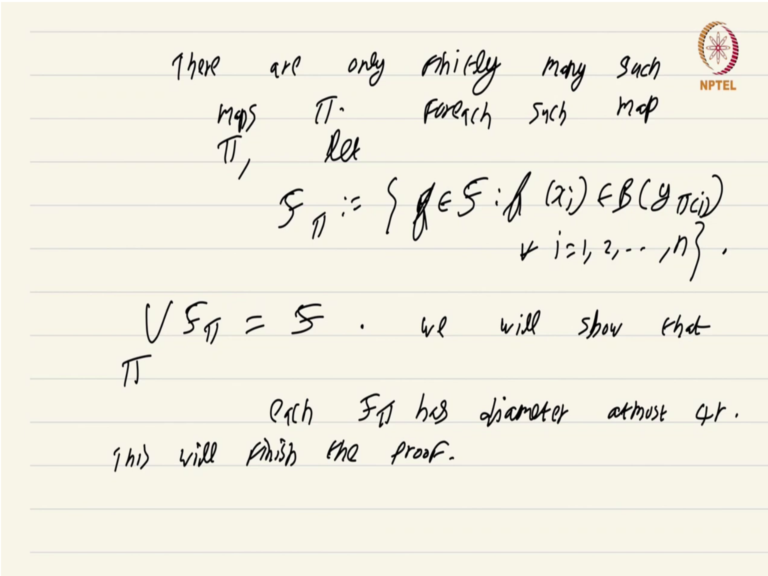
And, this is true for each F. Given any element small f from script F, if you consider F x 1 to F x m, they have to be contained in one of these, but we do not know which of these they are contained in ok.

Now, what we do is here is the part where we pull a clever trick, to make this precise that F x 1, F x 2, F x n is there in b y 1 to b y m. What we can say is there is a map, there is this map capital pi, capital pi from 1, 2 dot dot dot n to 1, 2 dot dot dot m, such that F of x i is an element of B of pi of i ok. Sorry, B of y subscript pi i.

Think about this carefully for a second. This is just making precise the fact that, F x 1 will belong to 1 of B y 1, B y 2, B y m, it could be 1 it could be more also, I am not denying the possibility. All I am saying is there is a map pi from 1 to n to 1 to m, such that F x i will be in the ball, y B of y subscript pi of i ok.

So, just pass this for a few minutes and you will definitely understand what is going on ok. Now, here is the key step. This pi map will change from function to function right. Given any function there will be a pi map is what we can conclude. There are only finitely many such maps right.

(Refer Slide Time: 30:45)

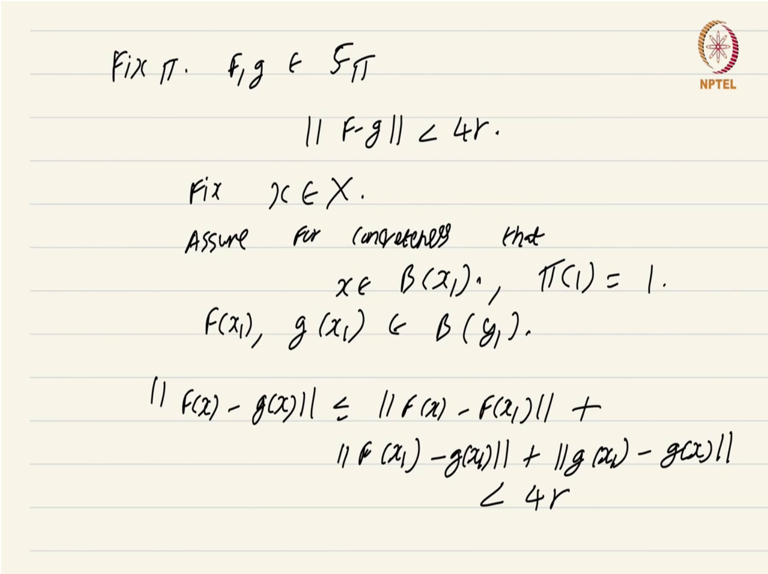



There are only finitely many maps from 1 to n to 1 to m, you can write down explicitly the number of maps also. There are only finitely map, finitely many such maps pi ok. For each such map, for each such map, each such map pi let F subscript pi be the set of all F in F, set of all F in F, let me for consistency again write this F in this really ugly hand writing. Such that, F of x i is there in B of y pi i for all i equal to 1, 2 dot dot dot n ok.

So; that means, I am going the other way around given an f there is a pi, that does the job for us, I am saying fix a pi and look at all those functions for which pi will do the job ok. And, I am putting that as F pi.

Now; obviously, union of F pis is equal to F as you run through the maps pi. You look at all the possible maps from 1 to n to 1 to m and you take their union so; obviously, it will be equal to F ok. Now, what we are going to show is that, we will show we will show that each F pi has diameter at most 4 r ok. If, you can do this will finish the proof. I will leave the rest of the details to you, it is rather easy, we will show that each such F pi has diameter at most 4 r.

(Refer Slide Time: 33:01)



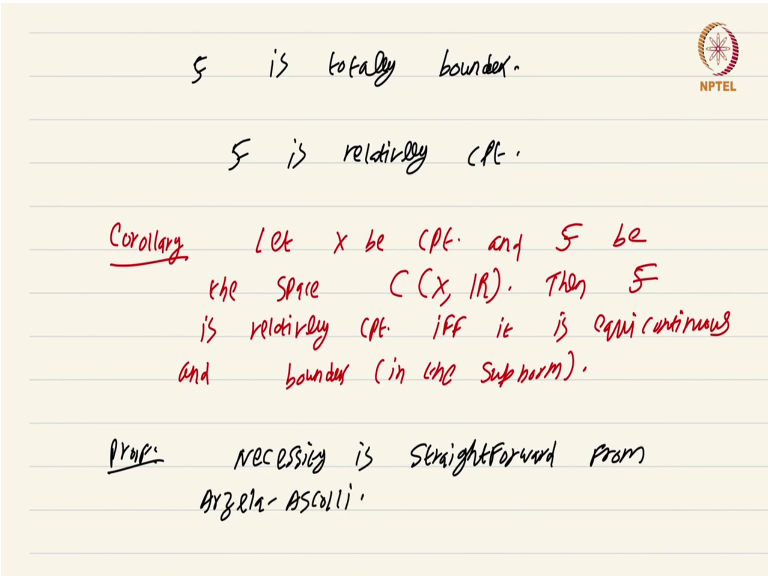

So, what must be do? Take F comma g coming from this F pi you have fixed a pi take F comma g, coming from this F pi. What we must do is we must show that norm F minus g is less than 4 r right. That will show that the diameter of F pi is less than 4 r or less than or equal to 4 r ok. Now, fix x in x also, fix x in x also, what we are going to do is, we are going to try to show that norm F x minus g x is less than 4 r that will finish the job for us.

So, assume for concretenessthat x is in the very first ball B x 1. Recall we had begun by choosing fixing an r and choosing n points x 1 to x n such that, B x 1 union B x 2 union dot dot dot B x n is a cover of x ok.

Now, what I am saying is without loss of generality, we are going to assume just for the sake of concreteness, that this given point x in B x 1 the argument will not depend on this particular index at all. And, we are also going to assume that pi of 1 equal to 1.

Again it really does not matter; this is just for our convenience ok. Then, what does this mean? This just means F x 1, comma g x 1, are both there in B y 1 that is what this means right. Now, we just compute norm F x, minus g x is less than or equal to norm F x, minus F x 1, plus norm F x 1, minus g x 1, plus norm g x 1, minus g x ok. And, this is certainly going to be less than 4 r, this is certainly going to be less than 4 r. And, this finishes the proof this finishes the proof that F is totally bounded.

(Refer Slide Time: 35:33)

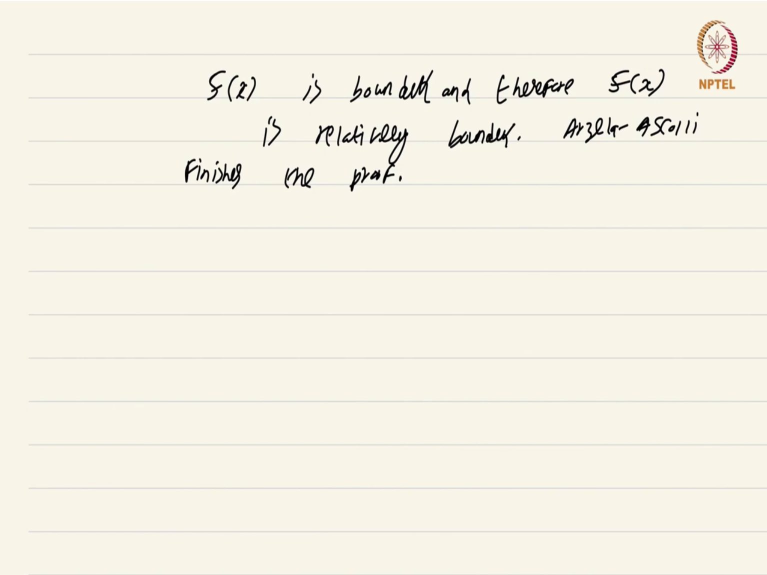



So, this we can conclude that F is totally bounded. And, from the results that we have seen on the section on compactness it follows that F is relatively compact, F is relatively compact. So, this concludes the proof of Arzela Ascolli theorem, there is another proof of Arzela Ascolli theorem that repeatedly you choose subsequences and do a diagonal argument; I am going to leave that as an exercise to you ok.

Let me just end with 1 corollary of the Arzela Ascolli theorem, which is the most useful version of Arzela Ascolli in real life. Let X be compact and let F be the space C X R ok, I could choose B C X R, but it is the same because X is compact. Then, F is relatively compact is relatively compact, if and only if, it is equi continuous, it is equi continuous and bounded in the sup norm ok, note not totally bounded ok.

Now, proof necessity is straightforward from Arzela Ascolli, I want you to check this, necessity is straightforward from Arzela Ascolli ok. Now, for sufficiency we know that, we have to show that, F is equi continuous. And, we also have to show that each orbit is relatively compact. We are already given equi continuity and boundedness in the sup norm, we have to show, that each orbit is relatively compact.

(Refer Slide Time: 37:55)




Now, each orbit F x is of course, bounded, each orbit is bounded simply because we are assuming that the family F itself is bounded in the sup norm ok. And, therefore, it is going to be relatively compact. Simply, because we are in r therefore, F x is relatively compact. Here is a crucial point where we require, the fact that the co domain is r, it is not true in general that a relatively I mean a bounded set is going to be relatively compact.

Now, again Arzela Ascolli finishes the proof, Arzela Ascolli finishes the proof. So, this version of Arzela Ascolli is one of the most commonly used ones where the co domain is r. So, this concludes this somewhat detailed and technical section, it is better that you actually go through the proof by yourself, rather than hearing me do it is somewhat involved proof. So, please go through it carefully and work out the exercises also.

This is a course on Real Analysis and you have just watched the video on the Arzela Ascolli Theorem.