Real Analysis II

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Lecture - 7.3
Connected and Path-Connected Components

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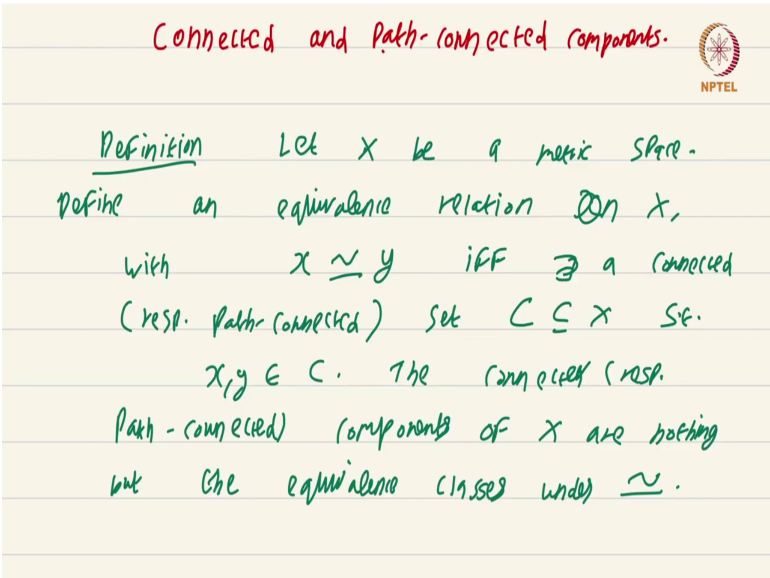

Often the metric space that we are considering will not be connected; but the assumption of connectedness might simplify our study. Thankfully, if a space is not connected, it always breaks up into pieces that are connected and in such situations we can deal with each piece separately to simplify our analysis. So, let us define what these Connected Components and Path-Connected Components are which are supposed to be the pieces.

Definition let X be a metric space. Let X be a metric space. Define an equivalence relation x on X with x being related to y if and only if, there exists a connected respectively path connected.

So, this is going to be a dual definition; both the definition of a connected component as well as a path connected component will be built into this definition; if and only if, there exist a connected respectively path connected set C x or let us just say C. No need to complicate the notation any further; connected set C subset of X such that x comma y are both in C ok.

So, what this says is two points are related if you can find a set C that contains both the points and C is connected, that is for the definition of connected component and C is path connected for the definition of path-connected component ok. Now, the fact that this is an equivalence relation is an very very easy exercise.

You have already shown that the union of connected or path-connected sets that contain a particular point in common is always going to be connected respectively path connected, using that you can show that this is an equivalence relation quite easily.

So, the connected respectively path connected components of X components of X are nothing but the equivalence classes under this relation.

So, the space breaks up into various pieces and it will turn out that each pair of points that is there in a given connected component, a given path connected component can be connected by a path and each connected component itself is going to be connected ok. Let us prove this fact because that is what this definition is actually modelled on, what we I mean this entire definition is to make sure this proposition is true.

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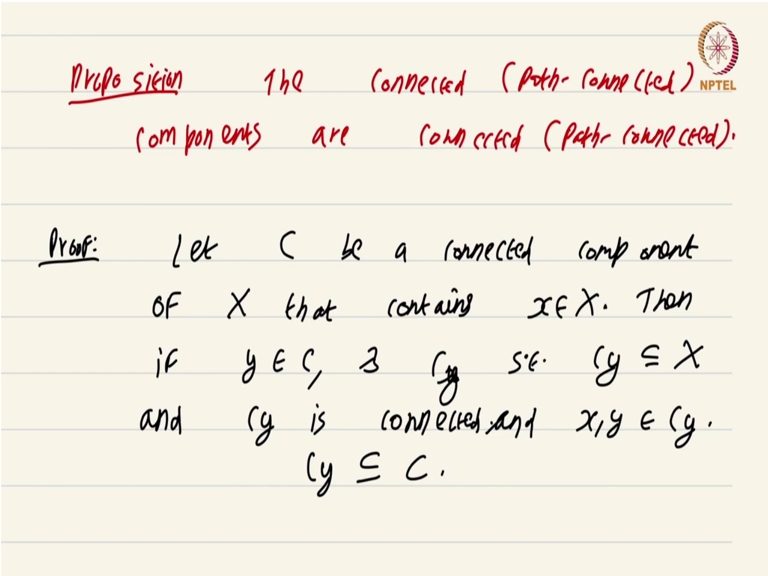

So, let us just first prove this proposition. The connected path-connected components are connected path-connected. So, the whole definition was to ensure that this is true ok. Now, the proof is very very easy. So, I will just consider the case of the connected components. Proof, let C be a connected component; be a connected component of X that contains x a given point. Then, if y is in C, there exist some C x or rather C y such that C y is a subset of X and C y is connected ok.

Of course, there exist C y such that C y is subset of X and C y is connected and of course, I must mention x comma y are both in C y. So, given any given any point x in X we can and a point y in C, where C is the connected component that contains x, then we can always find a connected set C y that contains both x and y. Now, it is obvious that in fact C y is a subset of C. This just follows because of the very definition of the equivalence relation.

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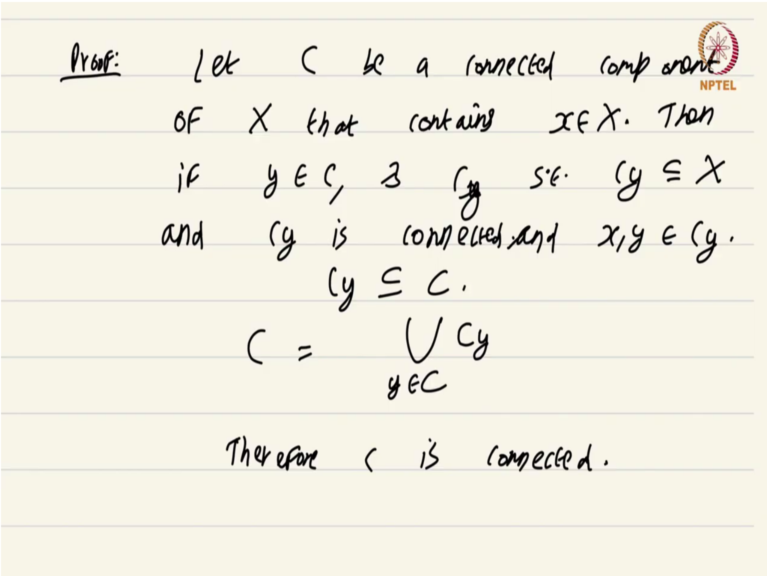

In other words, C is nothing but the union of all C y as y runs through C ok and all these C y's contain the point x that is how Cy was defined. But a union of connected sets that contain a point in common is connected; therefore, C is connected. Therefore, C is connected. Since you can express the connected component that contains X as a union of connected sets all of which contain X, the union is going to be connected ok.

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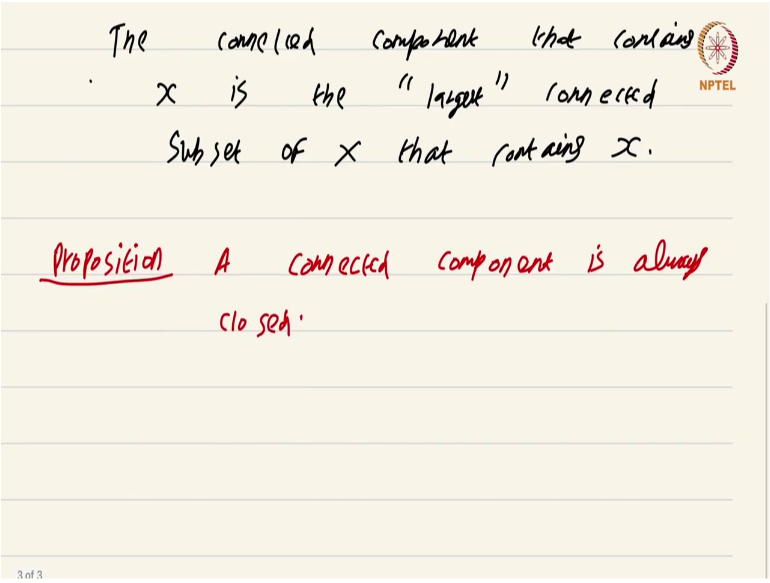

So, this proposition prompts the following remark. The connected component; the connected component; that contains x; that contains x is in some sense the "largest" connected subset connected subset of X that contains x. So, it is a piece. It is one of the pieces of x and it is the largest piece that contains the given point x ok. Now, I am just going to prove a couple of basic facts about connected components.

These connected components will become more important in a course on topology or manifolds or some such. So, deeper properties, you will probably learn; there is not much more to this notion. So, first is a connected component is always closed; connected component is always closed and the proof of this is just one line.

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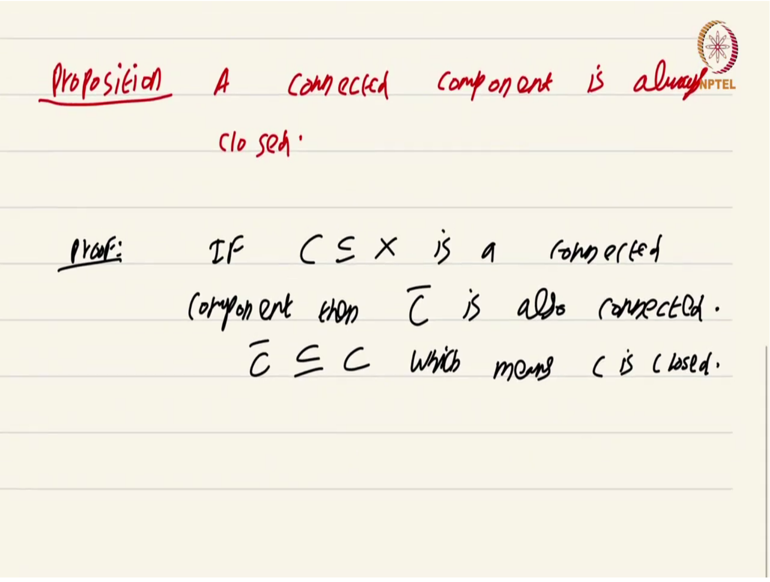

Proof is if C subset of X is a connected component; is a connected component, then C closure is also connected; C closure is also connected right. So, given the fact that C is a connected component; that means, that if you take any point x in C and if there is a point y, such that x and y belong to the same connected set, then that entire connected set should belong to C.

So, applying this argument C closure which is connected immediately shows that C closure must be a subset of C, which means C is closed, which means C is closed. So, this in some sense is following because connected components are in some sense definitionally the largest connected species of the set x and C closure is a larger connected piece ok.

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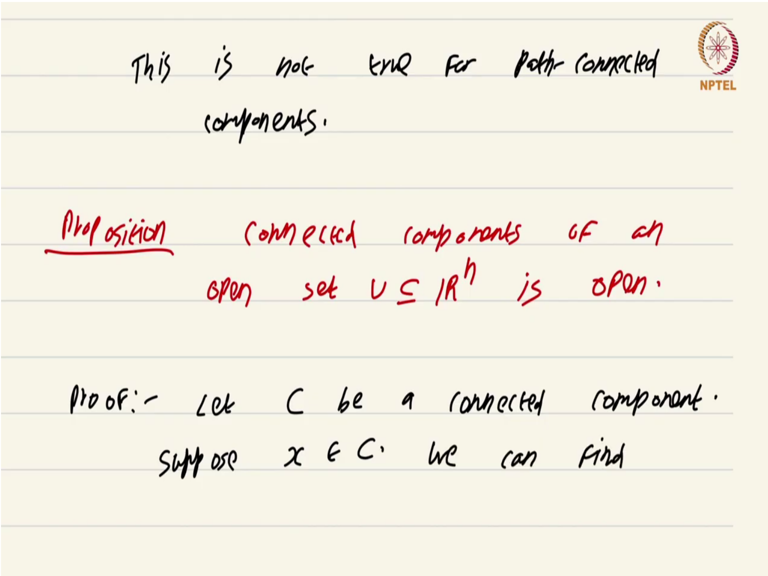

So, here is another remark needs to be made, this is not true. This proposition this is not true for path-connected components; for path-connected components. So, please come up with a counter example to see why this is not true for path connected components. In fact, we have already seen a counter example. It is more like you have to figure out which is the counter example.

So, let me end this short video on connected components with this proposition. Connected components of an open set open set U in R n is open ok. Proof and the again, the proof is rather easy. Let C be a connected component. Let C be a connected component. Suppose, x is in C.

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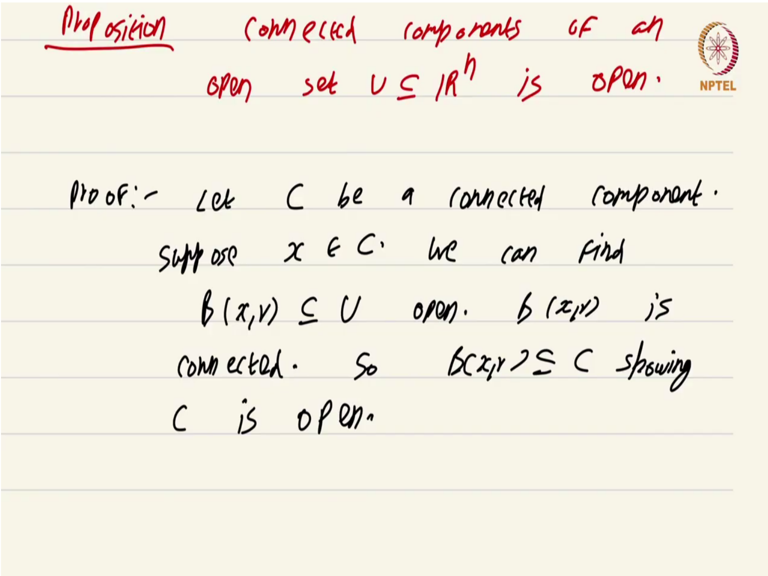

Then, by openness, we can find we can find B x, r subset of U open right, open ball and b x, r is connected. I want you to think about why b x, r is connected? We have actually shown it already b x, r is connected. So, b x, r must be contained in C, showing C is open; showing C is open.

Now, let us end this section on connectedness with the remark, you might wonder why is this not true for an arbitrary metric space; I mean after all we have used is the fact that a given any open set, we a given a point, we can find a ball. Well, the reason is b x, r need not be connected in an arbitrary metric space. It is connected in R n because it is actually convex.

So, I am going to spoil the surprise for you. It is actually connected because it is a convex. This is not in general. The notion of convexity itself does not make sense in any arbitrary metric space. Furthermore, you can construct examples of metric spaces, where b x, r is not going to be connected. This is a course on Real analysis and you have just watched the video on connected components.