Real Analysis II
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Lecture - 7.2
Path-Connectedness

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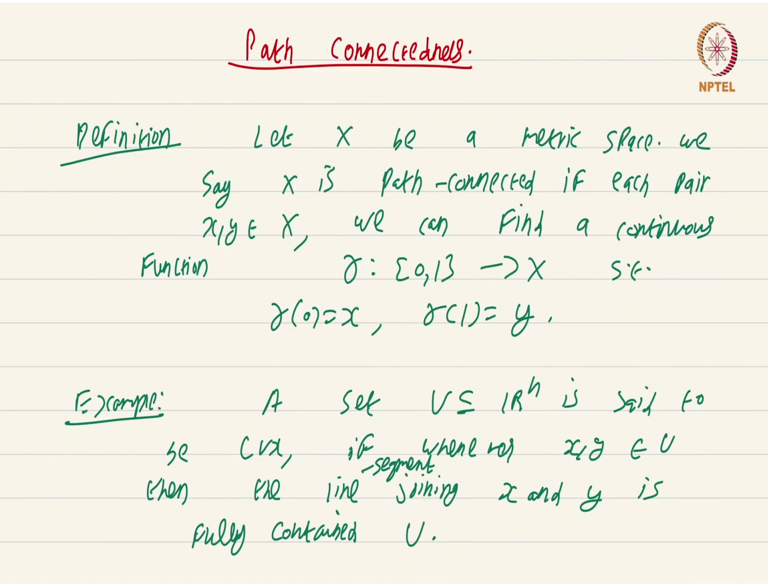

The notion of connectedness is not very easy to check in general. Thankfully, there is a stronger notion called path connectedness that is often much easier to check. And not only that path connectedness, usually these spaces be considered that is manifolds are connectedness and path connectedness coincide for such spaces.

So, let us begin with the definition of path connectedness. It is not unintuitive, it is straightforward. The definition is as follows. Let X be a matrix space; let X be a matrix space.

We say X is path-connected; X is path-connected. Let me just remark that path connectedness is also referred to as arc wise connectedness by some authors, we say X is path connected if for each pair x comma y in X.

We can find we can find a continuous function gamma from 0, 1 to X. So, such functions are called paths that is a continuous function from the closed interval 0, 1 to X again there is nothing special about 0, 1. You could have taken any close interval it would have worked.

So, we can find a continuous function gamma from 0, 1 to X such that gamma of 0 is equal to x and gamma of 1 equal to y. In short, given any two points in the metric space X, we can find a continuous path that connects x and y. So, the terminology is very geometric and visual.

Let us see an example; one example. In fact, I will give you an entire class of examples. A set U subset of R n is said to be convex; is said to be convex, if whenever x comma y is in U, then the line joining x and y; line joining x and y x and y is fully contained in U. Rather the line segment is a better way to describe it. The line segment joining x and y is fully contained in U.

So, it is almost by definition that convex subsets of R n are all path-connected, it is straightforward it just come from the definition. Now, to your hearts content you can draw many many convex subsets of R n to see that there are plenty of examples of path connected sets.

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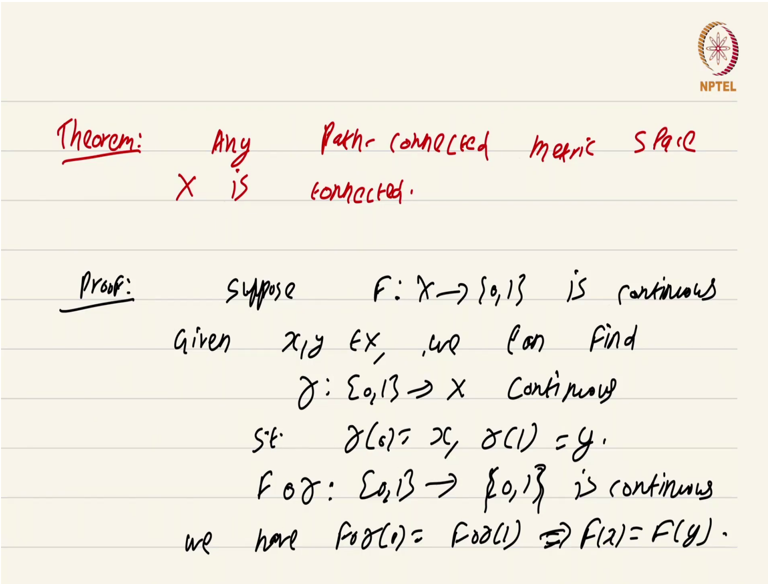

Now, coming to the important fact that we need, the fact that path-connected spaces are automatically connected and this theorem is very easy to prove. It is in fact, just one line theorem. Any path connected metric space; path connected metric space X is connected and the proof is literally just a line long. Proof: suppose, F from X to 0, 1 is continuous.

Suppose, you take a continuous function from the space X to 0, 1. Given, x comma y in X, we can find we can find gamma from close 0, 1 to X continuous; such that gamma of 0 is x, gamma of 1 is y this is just the definition of path connectedness. Then, all you do is compose F o gamma. So, F o gamma from 0, 1 to 0, 1 is continuous.

So, on the right-hand side when I said 0 1 I mean the closed interval 0 1 on the sorry, on the left-hand side, I meant closed interval on the right-hand side it is just the set comprising the two elements 0 and 1. But because 0, 1 is connected we must have we have F o gamma of 0 is equal to F o gamma of 1 which is equal, which just means F of x equal to F y.

So, in other words, we have shown that the function F is constant. So, this quickly disposes of the proof that any path-connected matrix space is connected.

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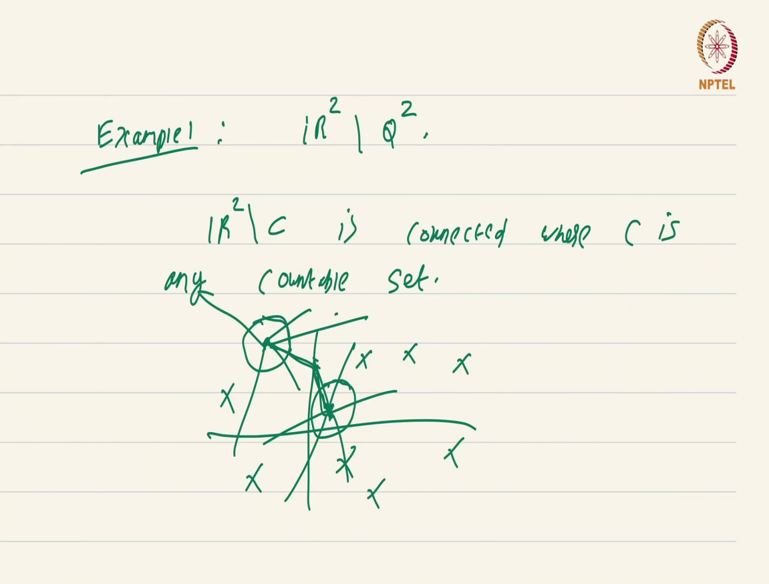

Now, let us see some concrete scenarios, where we are able to show connectedness of some complicated spaces by exploiting path connectedness. Example 1 is R 2 minus Q 2, that is you take the plane and punch away all the points whose both co-ordinates are rational.

Now, if you just look at it, it looks like what I do not know what to say a piece of cloth sufficiently zoomed in. If you look at a piece of cloth that is sufficiently zoomed in, it will sort of look like R 2 minus Q 2. Now, this intuition sort of suggests that it is a connected set, but there is no easy way to show it just from using the definition of connectedness or the equivalent properties of connectedness, but you can immediately show this by path connectedness.

Now, I will just sketch a geometric proof and leave it to you to check that this can be made mathematically precise. What you do is the following. In fact, what I am going to do is, I am going to show that R 2 minus C is connected, where C is any countable set. It does not matter, there is nothing special about R 2; I mean about Q 2 is any countable set.

So, the key fact is that we are removing only a countable set that is the key. So, what you do is you have this R 2 and lots of points are punched out ok. Now, take two points that you are interested in, you want to find a path in between them. What you do is you draw small circles around these two points ok. Once you have done that observe that this circle has uncountably many points. Prove that it is rather easy to show.

A circle will have uncountably many points, same is true of this circle of course, that means, if you consider the various rays starting from these points and passing through these circle, there will be uncountably many such rays similarly here, there will be uncountably many such rays starting from the two points.

But we have removed a countable set just C is a countable set; that just means, that from these uncountably many I mean if you consider these uncountably many rays. Only a countable sub collection of them will contain points of C right, because C is countable and you have uncountably many rays.

Now, elementary geometry will just show you that you will be able to find two rays that actually intersect. And these two rays will not contain a single point of C ok and therefore, we have found a path. In fact, we have found a path that comprises just two-line segments that connects these two given points.

So, this is just an intuitive argument. I want you to make it precise that R 2 minus Q 2 is in fact, path connectedness is path-connected ok.

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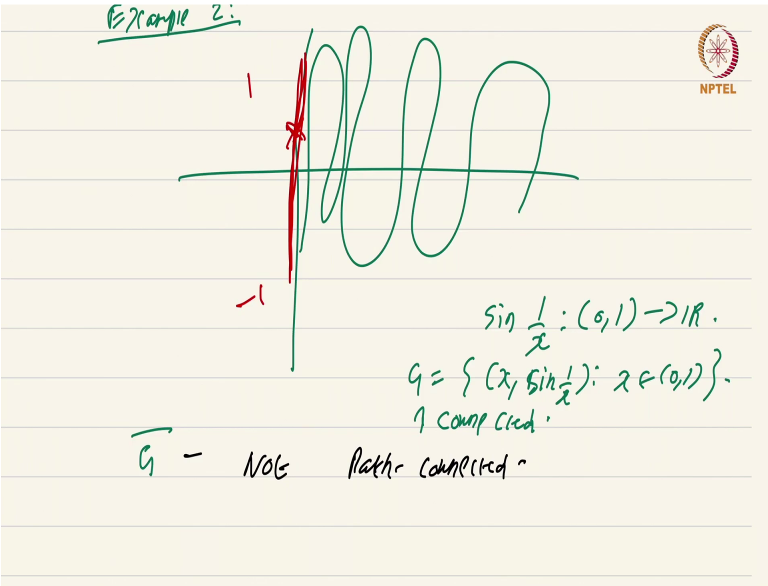

Example 2 is a nice one. It is to show that there can be connected sets which are not path-connected ok, connected sets which are not path-connected. Now, in an earlier exercise, you would have shown that the graph of any continuous function is automatically connected.

If you have not shown this, please do it now, ok. Now, what you do is you consider the topologist's sine curve. Again, this is a bad picture. This is this example more than being an illustrative example in topology, has been an illustrative example of why my drawing skills are nonexistent. Nevertheless, assuming your minds either this is the topologist's sine curve ok.

So, what I am going to do is I am going to consider just sin 1 by x from 0, 1 to R ok. I am just going to consider sin 1 by x from open 0, 1 to R. Now, the graph that is just the set of all points x comma sin 1 by x; such that, x is coming from 0, 1 this graph is connected; this is connected. Please show this, show that the graph is connected ok.

So, this graph which is lying in R 2 of sin 1 by x is connected. What I am going to do is I am going to consider G closure. Again, we have shown that any set that squeeze in between a connected set and its connected sets closure they are all connected, in particular the closure of a connected set is connected.

So, if you think for a while, you will notice that the closure of the graph of the topologist's sine curve, the way we have defined it, is just this graph along with this line vertical line segment from minus 1 to 1; that will be the closure ok. So, the closure of the graph of the topologist's sine curve, the way we have defined it is just the graph of the topologist's sine curve along with this vertical line segment on the y-axis from minus 1 to 1 ok.

Claim is that this G closure this is not path-connected; this is not path-connected. Now, this is sort of intuitively clear. If you start at a point somewhere on this axis, there is no way to jump to the graph. There is no continuous path that connects the points in this particular line segment to the graph of sin 1 by x. So, that is the basic idea. Let us establish this that rigorously.

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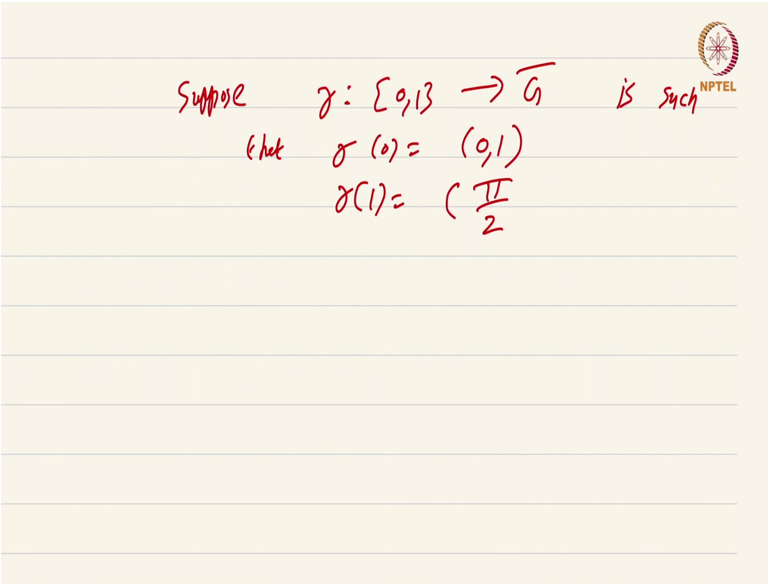

How do you do that? Well, let us take, suppose gamma from close 0, 1 to G closure is such that gamma of 0 is the point 0, 1, which is there on that vertical line segment and gamma of 1 is the point on the topologist's sine curve, which I am going to just take, it really does not matter what point it is. I am going to take for instance pi by 2 comma I will not take pi by 2, because at pi by 2 I do not know what value it is.

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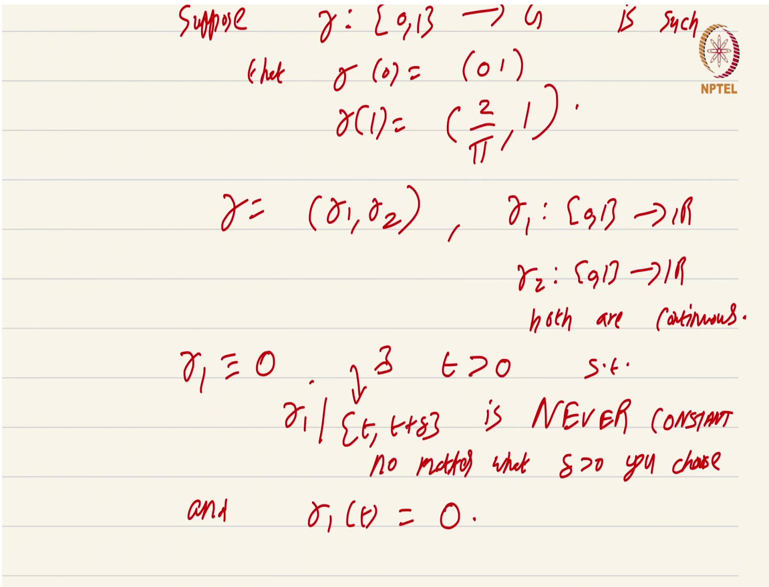

So, I will take 2 by pi comma 1 right. Sin pi by 2 is 1. So, sin 1 by 2 by pi is sin pi by 2 which is 1 ok. So, suppose you can connect gamma, you can connect 0, 1 and 2 by pi 1 with a path ok. Now, what I am going to do is, I am going to write gamma as gamma 1 comma gamma 2, write it as 2 components ok.

And by characterization of continuity into product spaces, both gamma 1 from 0, 1 to R and gamma 2 from 0, 1 to R; both are connected, both are connected not connected both are continuous both are continuous maps ok.

Now, what is happening is, eventually gamma 1 is traversing all the way from 0 to 2 by pi. So, it has to I mean this gamma 1 cannot be a constant mapping. So, what I am going to show is that gamma 1 is actually identically 0, which leads to a contradiction ok. Now, what I do is, since I am assuming gamma 1 is not constant there exist.

So, this is the crux of the proof and as a very good teacher, I am going to leave the crux to you. There exist t greater than 0 such that gamma 1 restricted to t comma t plus delta ok is never constant, no matter what delta greater than 0 you choose ok and gamma 1 of t is 0 ok.

So, please prove this claim. There is a t such that, when you restrict gamma 1 to t comma t plus delta, it is never going to be constant, no matter what delta greater than 0 you choose. Now, this looks very complicated, but let me reassure you that it is not so difficult to show this.

Essentially, gamma 1 is traversing from 0 to 2 pi. So, at some point it should start. It is just capturing the fact that gamma 1 should start moving, it cannot just sit there at 0 forever. It has to start. I am calling the point where it starts t ok there must exist a point. Again, another hint.

Try to prove this by contradiction. If there does not exist such a point, then show that gamma 1 has to be identically 0 ok, now. So, this gamma 1 from t to t plus delta is never constant, no matter what delta you choose. Now, once we have gotten this t, such a point where this curve gamma 1 not curve, this function gamma 1 is eventually starting and not just staying put what you do is the following.

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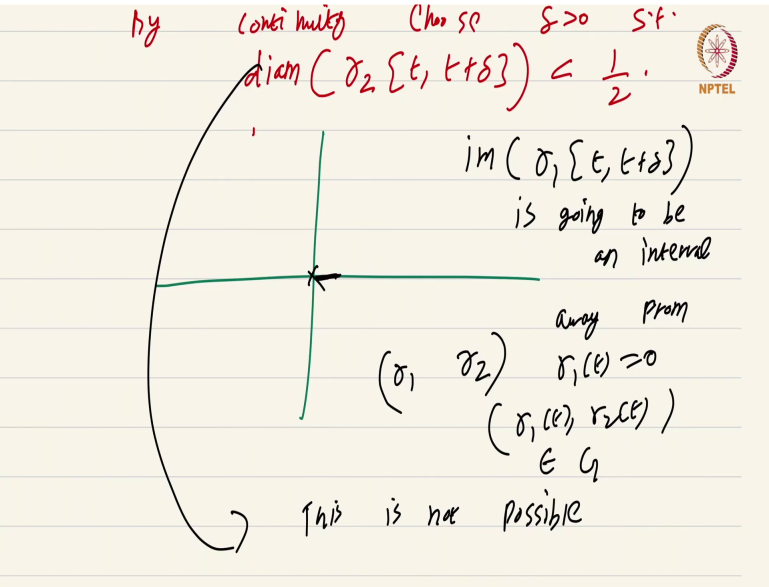

By continuity choose delta greater than 0 such that diameter of gamma 2, we forgot gamma 2 it is feeling a bit lonely. So, look at the image of gamma 2 from t to t plus delta. Choose this to be less than half. You can obviously do this by continuity, gamma 2 is also a continuous function, but wait a second something weird is happening.

What did we what did we do? A picture in this scenario is really going to be worth a thousand words. What is happening is gamma 1, no matter what delta you choose its going to move right. It is the image of gamma 2, image of gamma 1; not gamma 2 image of gamma 1 of t of t plus delta is going to be is going to be an interval. In fact, it is going to be a closed interval.

And one end point of this interval is going to be 0, because we know that gamma 1 of t is 0 it is going to be a non singleton interval right. It is not going to be a singleton interval. Why is it not going to be a singleton interval? Because we chose this t as the point, where gamma 1 finally starts moving right.

So, image of gamma 1 t t plus delta is going to be some interval of positive length ok, which means that if you look at gamma 2; gamma 1 comma gamma 2 have to be points on the graph of the topologist’s sine curve away from t equal to 0, whenever gamma 1 of t is not sorry, away from gamma 1 of t equal to 0. We know that gamma 1 of t gamma 2 of t is there in the graph which we call G; if I believe is there in the graph of the topologist’s sine curve ok.

But here, is an issue. We are at an interval that is touching 0, a interval like this. We know that as you approach 0 the topologist’s sine curve starts oscillating rapidly ok. So, this fact that diameter of gamma 2 t comma t plus delta being less than half; this is untenable this is not possible; this is not possible.

The graph I mean the diameter of gamma 2 t comma t plus delta being less than half is simply not possible, if gamma 1 is going to take values here. So, this contradiction shows that you could not have found a path gamma starting from 0, 1 and ending at 2 by pi 1. There is nothing special about the second point. There is a way to rewrite this proof in an obvious way, which does not involve or prove by contradiction. Just show that any path that starts at 0, 1 has to be constant that is also a good proof ok.

So, here is another example, where we see path connectedness playing a role, but path connectedness is a stronger condition than connectedness. There are sets that are connected, but not path-connected. Thankfully, in the most interesting scenario both coincide. I mean of course, I said its true for manifolds, but we have not defined what a manifold is yet.

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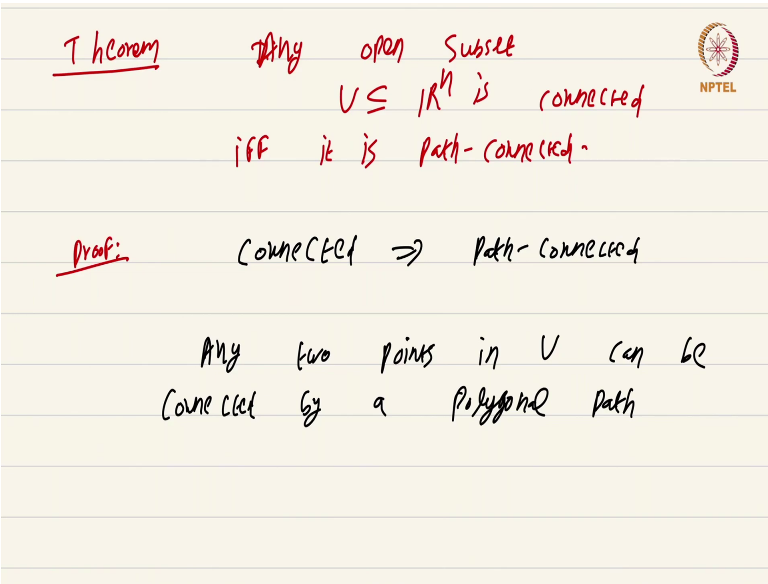

So, theorem: if any open subset; any open subset U of R n is connected if and only if it is path-connected, if and only if its path-connected. Proof: ok, one direction is true in general, path connectedness always implies connectedness. So, all we have to show is connected implies path connected ok.

And the technique I am going to use for this particular proof it might seem very obvious and straightforward once you understand it, but let me tell you that this technique that I am going to use is going to be used throughout mathematics. Whenever you want to show something about path-connected spaces, this is the technique that you are going to use ok. What we are going to do is we are going to show that any two points in U, any two points in U can be connected can be connected by a polygonal path.

What is a polygonal path? Well, it is a path which comprises of line segments. It is just a finite union of line segments ok. I am not going to make it precise, but you can do so using partitions. Essentially, what will happen is, you can partition 0, 1 in such a way, that on each interval determined by the partition, the image is just a line segment ok.

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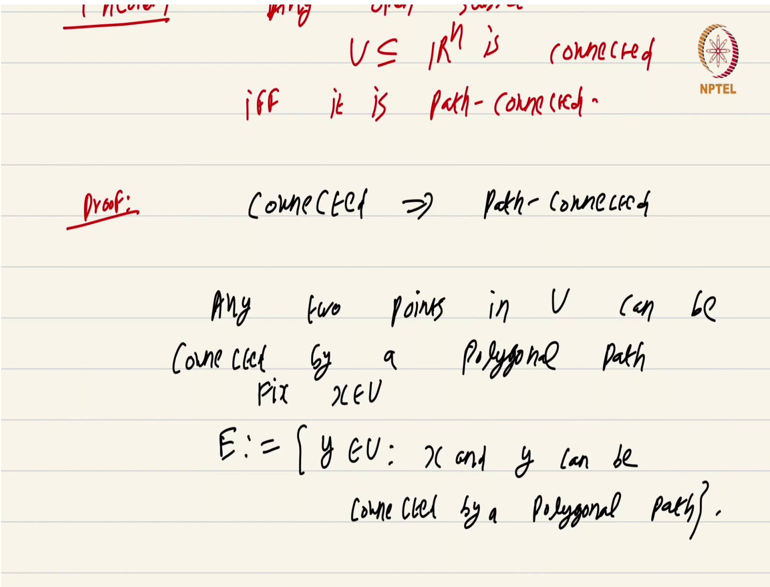

Now, what I am going to do is, I am going to define this auxiliary set, which is typically called E. This is the set of points y in U. So, first I will write fix x in U. Look at the collection of all points y in U such that, x and y can be connected can be connected by a polygonal path ok.

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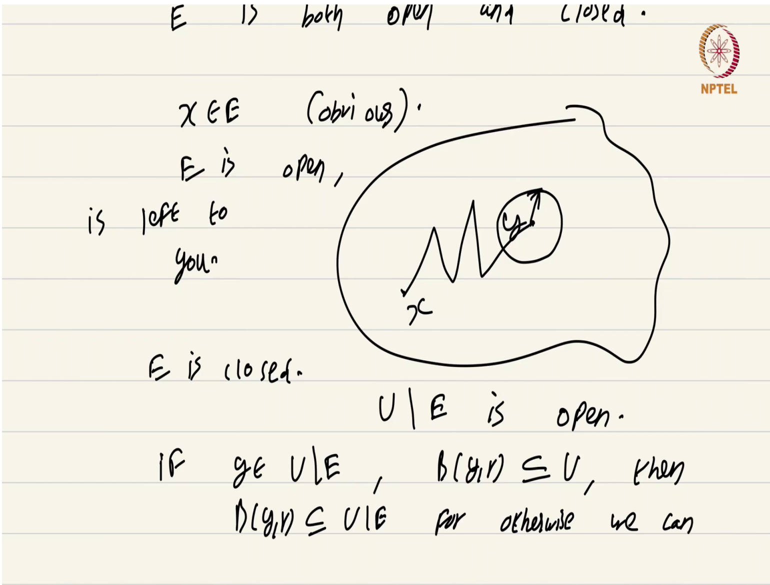

And what I am going to show is, I am going to show E is both open and closed ok. So, I hope you got the basic idea. Since, the set U is assumed to be connected, the only closed and open sets are going to be. Only closed and open subsets of a connected set are U itself and the empty set, but E cannot be the empty set; that is what we are going to argue.

So, first of all x is in E, this is obvious. So, E is non empty ok. Second of all, E is open. Why is E open? Well, because let us just draw a picture and I will leave the rigorous details to you it is rather obvious.

Suppose, y is there in E suppose this point. Just choose a ball or in this case yeah just choose a ball centered at this point y then; obviously, if you can join x and y by some polygonal path.

You can join every path in every point in this ball also by a polygonal path just travel to x from x to y by this polygonal path and then, just join one more line segment to it ok.

So, and the E is open is left to you ok. I mean essentially, just make this argument precise. Now, we will show E is closed. That is the last part and to show that we will show U set minus E is open and that is also equally easy, because if y is in U minus E; y is in U minus E and again, you take a ball B y, r, which is fully contained in U then, B y, r is contained in U minus E for otherwise by the exact same argument we can connect we can connect x to y by a polygonal path.

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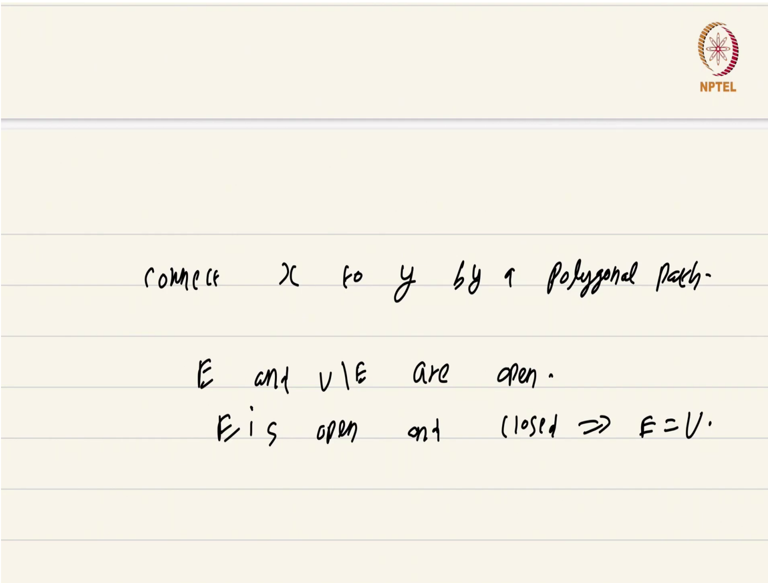

Remember, we started out with the point y, which is not there in the set E. What I am claiming is if U minus E is not open, then you can find some ball B y, r, which is fully contained in U. If this B y r is not fully contained in U minus E; that means, there is some point z in B y, r from which you can connect, x and z can be connected by a polygonal path just add another line segment you will be able to connect x and y ok.

So, this shows that both E and U minus E are open, which just shows E is open and closed; open and closed, but because U is connected this forces E equal to U; that just shows that any point can be connected to this given point x by a polygonal path right. So; that means, given any two points, you could have applied this argument to any two points and shown that they both will be connected by a polygonal path ok.

So, this proves the theorem in, for open subsets of Euclidean space, connectedness and path connectedness coincide. This is a course on real analysis and you have just watched the video on path connectedness.