Real Analysis II
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Lecture - 7.1
Connectedness

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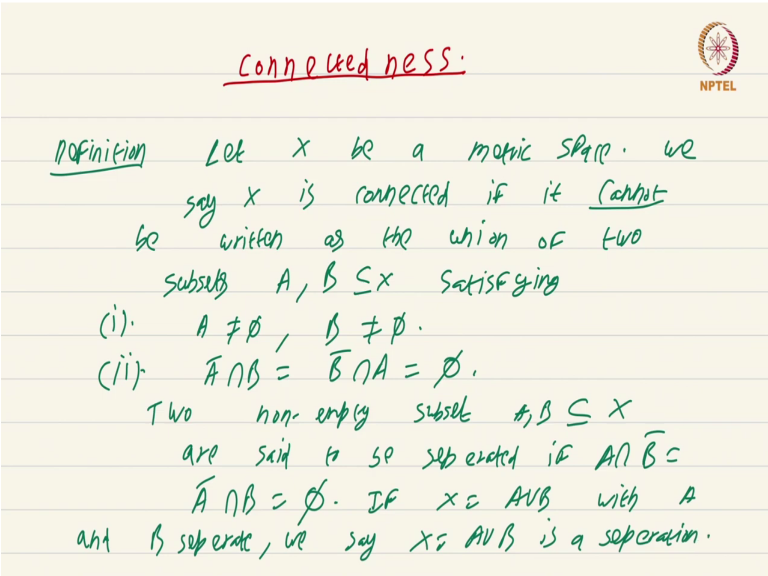

The notion of Connectedness for metric spaces is a bit more technical than the corresponding notion for the real numbers, but at the same time there is a lot of visual element to it; which we will discuss at length when we come to path connectedness. So, let us just define the notion of connectedness which is exactly the same as what we have already seen for metric spaces; for real numbers.

Definition let X be a metric space. We say X is connected; X is connected if it cannot be written; cannot be written as the union of two subsets; of two subsets A comma B of X satisfying, number 1: A is not the empty set B is not the empty set. 2: A closure intersect B equal to B closure intersect A is empty ok.

So, a set is set to be connected if you cannot write it as the union of two pieces that are not just disjoint, but in a sense essentially disjoint, that is what this A closure intersect B and B closure intersect A equal to empty saying. Of course, neither of these sets should be empty.

Now two non empty sets; two non empty sets subsets A comma B of X are said to be separated; are said to be separated if A intersection B closure equal to A closure intersect B equal to empty set. In other words, if you can write X as the union of separated sets then we say X is not connected ok. So, if X equal to A union B with A and B separated A and B separated we say X equal to A union B is a separation ok.

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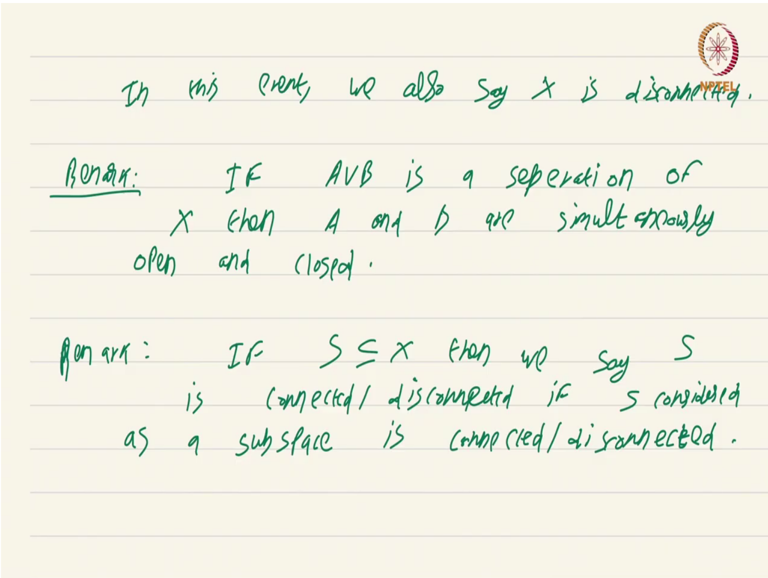

In this event we also say in this event we also say X is disconnected. So, the definition is exactly the same. Let me make a remark that will be utterly trivial to proof remark. If A union B is a separation of X then A and B are simultaneously open and closed.

I want you to check this it will take only a minute or two to check this, whenever you have a separation both these sets are actually going to be both open and closed. Another remark if S subset of X is I mean; is I mean if S is just a subset if S is just a subset then we say S is connected or disconnected; connected or disconnected if S considered as a subspace.

So, we treat S as a metric space in its own right is connected or disconnected, that is sort of the obvious way to formulate connectedness for subsets of a metric space. Now, note that the definition of connectedness did not actually involve the notion of the metric.

All it said was A closure intersect B and A intersection B closure should be empty and we have seen closed sets can be defined entirely in terms of open sets, closed sets are the duals of open sets that is the set is closed if and only if its complement is open. So, we do not have to go to the notion of metric to decide whether a set is closed or not we can just do it by considering the topology.

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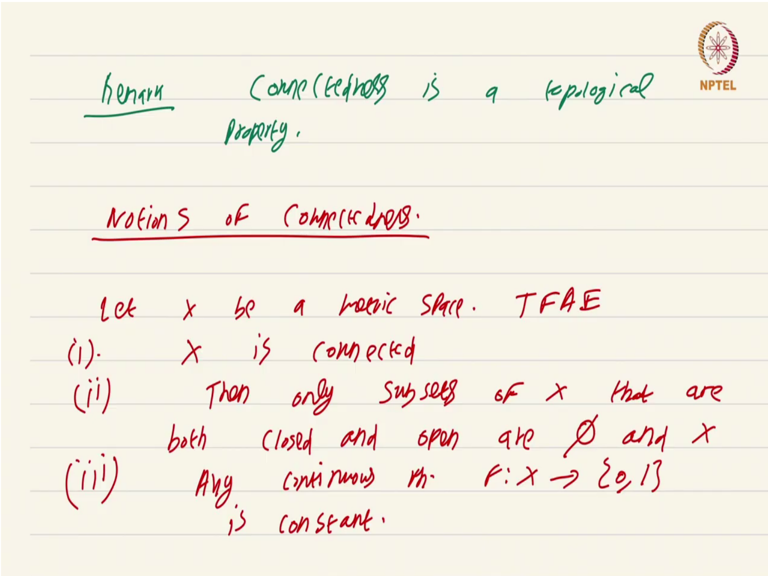

So, all this is leading to the final remark which is actually quite important connectedness is a topological property; connectedness is a topological property. What this means is that if you choose a metric space X D and choose another metric D prime that is actually equivalent to D in the sense that both D and D prime generate the same topologies, then X D is connected if and only if X D prime is connected.

So, connectedness is a topological property and it is some notion that does not depend on the choice of metric, but only on the topology that the metric generates. So, the moment we have a complicated notion like connectedness we should try to characterize it in multiple ways so that we understand it completely so notions of connectedness.

So, I am going to prove several statements are actually equivalent. So, let X be a metric space; let X be a metric space then the following are all equivalent. Number 1: X is connected. Number 2: the only subsets only subsets of X that are both; that are both closed and open are the empty set and X itself. Number 3: any continuous function F from X to 0 1 of course, you give 0 1 the discrete metric is constant.

I am not even going to bother writing down what the metric on 0 1 is. In fact, there is only one metric possible on 0 1 I mean essentially you can put many metrics, but all of them are essentially going to be the same as the discrete metric ok. So, any continuous function F from X to 0 1 is going to be constant.

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Proof (i) $\Rightarrow$ (ii).

S is open and closed. then

$X = S \cup S^c$

$\Rightarrow$ S^c is open and closed.

$\overline{S} \cap S^c = \overline{S^c} \cap S = \emptyset$.

If S and S^c are non-empty then X cannot be connected. $S = X$ or $S = \emptyset$.

(ii) $\Rightarrow$ (iii). Let $f : X \to \{0,1\}$ be continuous then $f^{-1}\{0\}$ and $f^{-1}\{1\}$ are both open and closed. $f^{-1}\{0\} = X$ or $\emptyset$ and

Again the proof of this is rather straightforward so let us begin by showing that 1 implies 2 ok. Now, suppose 1 is true; that means, if X is connected so suppose S is open and closed, suppose S is open and closed then of course, X is S union S complement and S complement is open and closed, because open and closeness are dual notions of each other. So, if you start with the closed and open set then its complement is also closed and open.

But because S and S complement are both open and closed S closure intersect S complement equal to S complement closure intersect S which are both empty. Which means if S and S complement are empty are nonempty then X cannot be connected; then X cannot be connected this will form a separation.

So, we are forced to conclude that either S is X or S is the empty set, which is what we had claimed ok. Now, coming to 2 implies 3. So, we assume that the only closed and open sets are those are the whole set X and the empty set let F from X to 0 1 be continuous. Then F inverse 0 and F inverse 1 the pre images are both open and closed; are both open and close.

This is because this singleton set 0 and singleton set 1 are both open and close subsets, because we are putting the discrete metric on the co domain. Inverse images of closed sets are close inverse images of open sets are open under continuous mappings are both open and closed ok. But you cannot I mean I mean the only possibility is that I; that means, F inverse 0 must be X or empty and F inverse 1 must be X or empty.

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$F^{-1}\{1\} = X$ or $\emptyset$.

This shows F is constant.

(iii) $\Rightarrow$ (i). Suppose $A \cup B$ is a separation of X.

$$F(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in B \end{cases}$$

Then F is continuous. But F is constant.

This shows $A \cup B$ could not have been a separation.

Because the only closed and open subsets are the empty set and the whole set this shows F is constant; this shows f is constant ok. Now, 3 implies 1 is also equally easy 3 implies 1 ok. What we are going to do is suppose A union B is a separation; is a separation of X. Now, we are going to define a function F of x is equal to 1 if x is in A, I put the flower brackets the other way round is 1 if x is in A 0 if x is in B ok.

Then F is; obviously, continuous then F is; obviously, continuous check that. Here it might be useful to use the sequential criteria or you can just take inverse images it does not matter you will be easily able to show that F is continuous. But; that means, F is constant this shows this shows A union B could not have been a separation could not have been a separation ok.

So, what we have shown is the notion of connectedness can be characterized in multiple ways.

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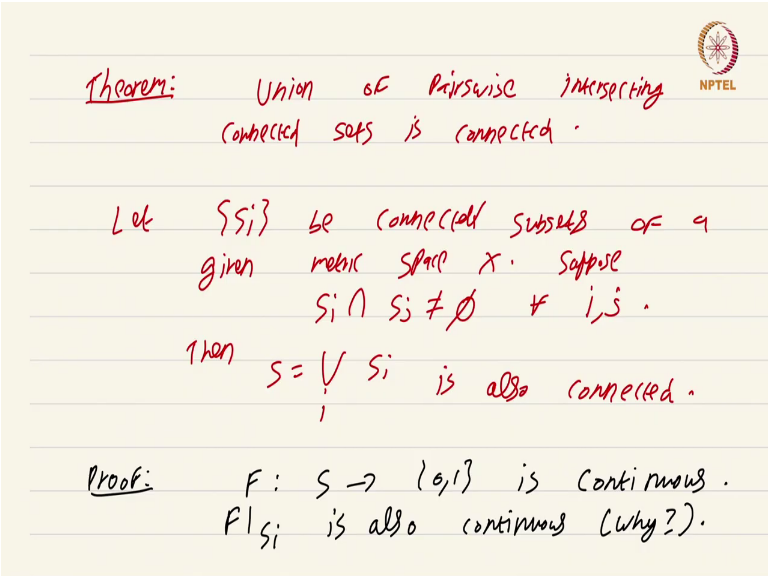

Now, it is advisable to use that formulation which is most apt for the scenario. We will illustrate this with a few theorems about connectedness. Theorem, union of pairwise intersecting connected sets is connected intersecting connected sets is connected. So, the setup is as follows let S i be connected subsets of a given metric space; given metric space X ok.

So, take connected subsets of a given metric space X. Suppose S i intersect S j is nonempty for all i comma j ok. Here the indexing set i it might indicate that it is the integers, but nothing like that it could be any indexing set you could take any collection it could be uncountable it really does not matter.

Then the union over i S i is also connected; is also connected. Now, there are several ways to prove this I am going to choose the way using the sequential not sequential the continuous function characterization of connectedness. So, let F so let us call this set S this union let us call this S.

Suppose F from S to 0 1 is a continuous function, the goal is to show that this function has to be constant ok. Now, because F from S to 0 1 is continuous F restricted to each S i is also continuous. Please check this is rather easy to do, but I am going to leave it to you as an exercise. The restriction of F to each S i is also continuous.

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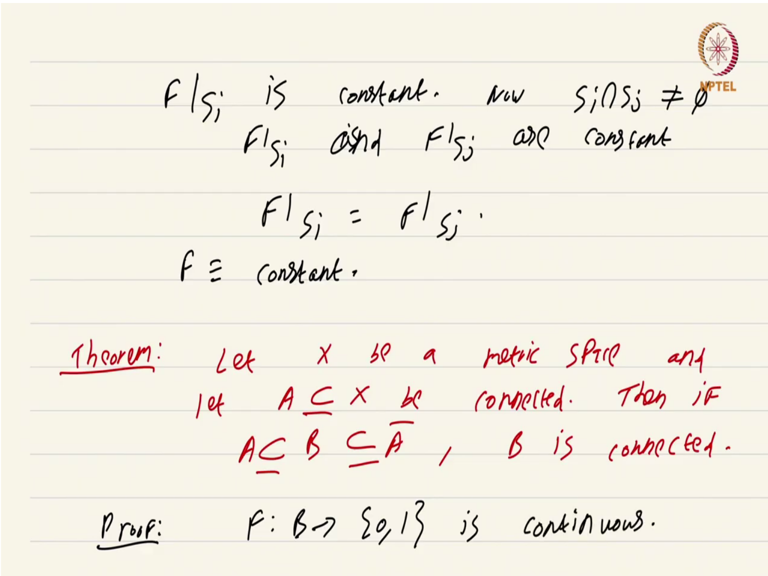

But this just means F restricted S i is constant, simply because each S i is connected ok. Now, S i intersect S j is non empty and F restricted to S i is constant S and F restricted to S j are both constant are both constant. Which just means F restricted to S i is equal to F restricted to S j, putting all this together you just get F is constant ok.

So, that was rather easy. So, I will prove another theorem which is along the same lines, I will try to prove as much as possible using the continuous version of connectedness. I urge you that you should try to prove it with these and in each scenario find out which is the easiest way to prove it.

Let X be a metric space; let X be a metric space and let A subset of X be connected be connected. Then if A is subset of B is subset of A closure; that means, take any set that lies in between A and A closure then any if A B is connected.

So, any set that is sandwiched in between a connected sets and its closure is automatically connected. Again here also the continuous version of connectedness the formulation in terms of continuous functions is very useful. So, suppose F from B to 0 1 F from B to 0 1 is continuous, suppose you take such a continuous function ok.

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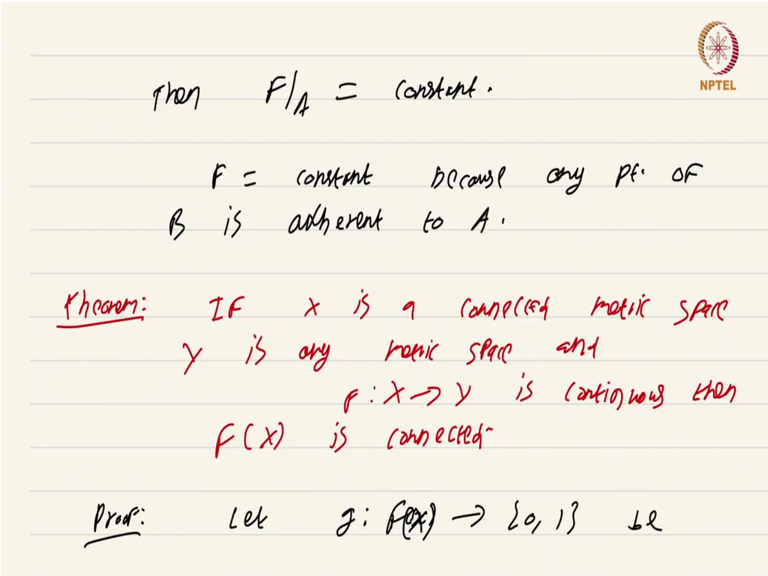

Then F restricted to A is of course, constant F restricted to A is constant. Which means F is constant as well; F is constant as well because, because any point any point of B is adherent to A.

This remark is not as cryptic as it looks like F restricted to A is constant take any point in B, then there is a sequence in A that converges to B as converges to this point in B therefore, the value at this point is also got to be the same value as the value of F which is just going to be constant either 0 or 1 ok.

So, its trivial to see that any set that is squeezed in between a connected set and its closure must be connected. Finally, this is the behavior of continuous functions under connectedness, if X is a connected metric space, if X is a connected metric space and Y is any metric space; Y is any metric space and F from X to Y is continuous then F of X is connected.

So, the continuous mappings take connected sets to connected sets this is actually rather immediate proof let g from the image F of X; F of X to 0 1 be continuous.

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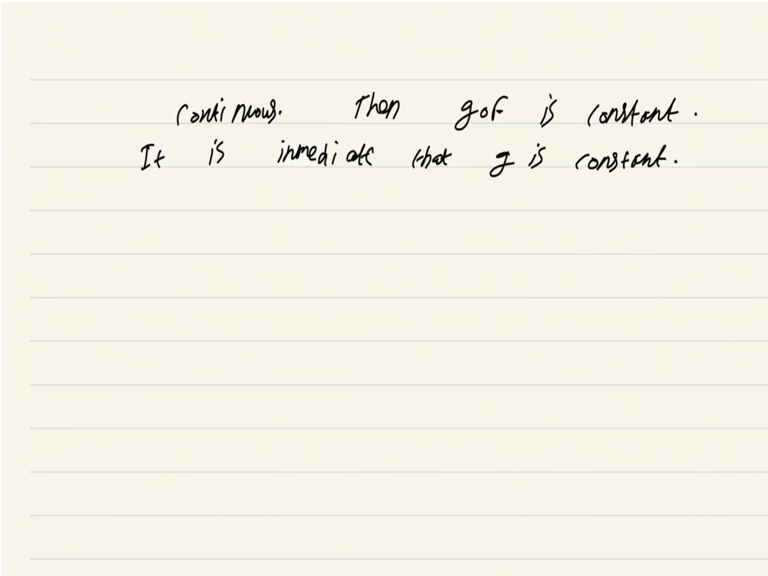

Then g circle F is constant because F is continuous g circle F is continuous so, g circle F is a continuous mapping from a connected matrix space so 0 1 therefore, it must be constant. It is immediate that g is constant from this immediate that g is constant. So, all of the results we have shown using the characterization of connectedness in terms of continuous functions to 0 1.

Think about how to prove these 3 theorems using the other characterizations of continuity and come to your other characterization of connectedness and come to the conclusion of which method is the best for the given scenario ok. So, this concludes this portion we will now go on to path connectedness and connected components in the subsequent videos. This is a course on real analysis and you have just watched the video on connectedness