Real Analysis II
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Lecture - 6.3
The Heine--Borel Theorem for Metric Spaces

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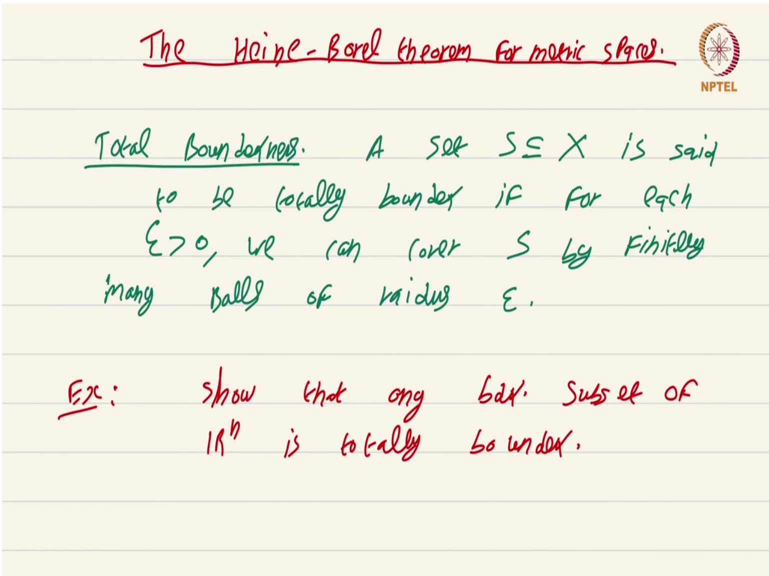

In this video, we are going to prove the Heine-Borel theorem for metric spaces. In the setting of real numbers, the theorem was rather straightforward a set is compact if and only if it is closed and bounded. We have already proved the same thing for R n. However, for general metric, spaces closed and bounded sets need not be compact. We have already seen couple of examples. So, the key fact we need in addition is completeness and a notion called total boundedness.

So, let us first define total boundedness. This is a much stronger requirement than being merely bounded. It says the following. A set S subset of a metric space is said to be totally bounded is said to be totally bounded if for each epsilon greater than 0, we can cover we can cover S by finitely many balls of radius epsilon. Finitely many balls of radius epsilon will cover the set S and this can be done for each and every epsilon; however, small it is, it does not matter.

So, I must give you an example, but instead I will give you an exercise. Show that any bounded subset of R n is totally bounded is totally bounded ok. Now, what we are going to do is prove the Heine-Borel theorem for metric spaces. After I prove the theorem, I want you to think about non examples of totally bounded sets. And the hint is we have already seen examples of closed and bounded subsets of metric spaces with are not compact use them to construct these non examples.

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Heine-Borel theorem: Let X be a complete metric space. Then a set $K \subseteq X$ is cpt. iff it is closed and totally bounded.

Proof: Observe that K is complete.
Let $x_n \in K$. WLOG, we may assume that $x_n \neq x_m$ if $n \neq m$. For a fixed $k \in \mathbb{N}$, finitely many balls of radius k, say $B_k^1, B_k^2, \dots B_k^{m_k}$ cover K.

So, let us state the Heine-Borel theorem and prove it. Heine-Borel theorem let X be a complete metric space let X be a complete metric space then a set K subset of X is compact if and only if it is closed and totally bounded.

So, this is a stronger statement in the sense that I mean it is not a stronger statement, it is a weaker statement than Heine-Borel theorem in the sense that the requirements the hypothesis is stronger. So, if you strengthen the hypotheses, the result will in general become weaker. So, the strong form of the Heine-Borel theorem is simply not true for metric spaces.

And one more remark since it should be rather obvious to you I forgot to mention it any totally bounded set is automatically bounded that should be obvious to you. So, let us go with the proof one direction is rather simple, the other direction requires a bit of delicacy or a delicateness, a bit of delicacy makes no sense, a bit of a delicateness.

Now, first of all observe that K is complete observe that K is complete simply because it is a close subset of a complete metric space, and therefore, it is complete. Now, let x n be a sequence in K. We are going to show that it has a convergence of sequence and use the fact that compactness and sequentially compactness coincide for metric spaces.

Now, without loss of generality which I abbreviate as W LOG, we may assume we may assume that x n is not equal to x m if n is not equal to m that is just passed to a subsequence such that this is satisfied. If it is not possible to meet this requirement, then the result is already over I mean the proof is already over this x m will definitely have a convergence subsequence.

So, we are going to take the case where we can find a subsequence of x n and remove repeating terms, we will get a subsequence, we are just re-indexing it and calling it x n. This is just for convenience. The proof will become easier. Now, for a fixed K in the natural numbers in the natural numbers, finitely many balls finitely many balls of radius K, of radius K, say B 1 K, B 2 K dot dot dot B m K K cover K right.

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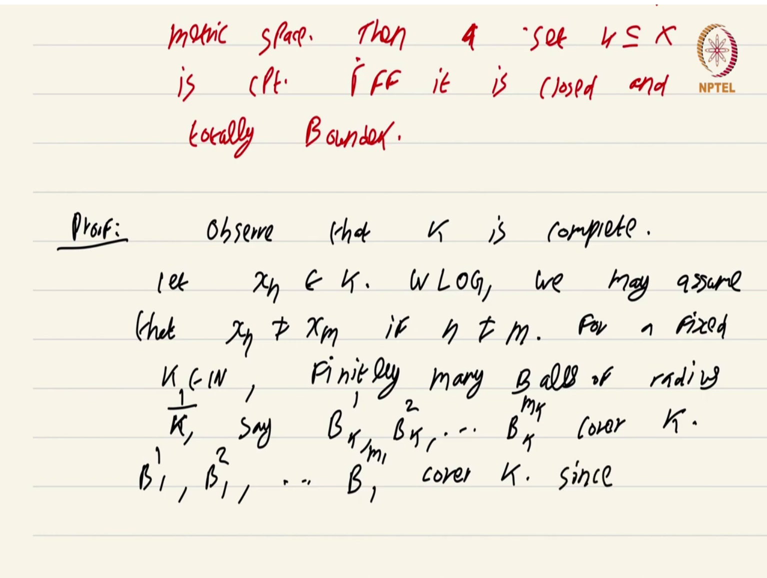

We are starting off with a totally bounded set. So, given any of radius 1 by K sorry about that ok. Given any fixed natural number K, we can find finitely many balls B 1 K, B 2 K, B m K K that cover K. And each of these balls are of radius 1 by K. Now, this m K in general will of course depend on K. You may have to increase the number. If you decrease 1 by I mean if you increase K, 1 by K will decrease you may have to increase m K. So, I am not claiming that this m K is in any way fixed or anything.

Now, suppose again this is just without loss of generality, suppose B 1 1 yeah sorry about that I skipped a step. So, we have this sequence of distinct terms x n and B 1 1, B 2 1 comma dot dot dot B m 1 1 cover K right from our notation.

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$\{x_n\}$ is infinite, infinitely many points of the set $\{x_n\}$ is contained in one of the sets

$B_1^1, B_1^2, \dots, B_1^{m_1}$ (pigeon-hole).

WLOG, assume infinitely many points of $\{x_n\}$ is in B_1^1. It is trivial to find a subsequence x_{n_k} s.t. $x_{n_k} \in B_1^1$.

Let $a_1 \in \{x_{n_k}\}$.

Since, this set x n is infinite, since this set x n is infinite, infinitely many points infinitely many points of the set of the set x n is contained in one of the sets one of the sets B 1 1, B 2 1 dot dot dot B let me just check back the notation B m 1 1, B m 1 1 ok. Simply because the set x n is infinite, infinitely many points of the set must be contained one of these sets this is just the pigeon hole principle again, this is just pigeon hole.

Now, without loss of generality, assume infinitely many terms infinitely many points of x n is in B 1 1 ok. Now, it is trivial, it is trivial to find a subsequence x n K such that x n K is in B 1 1 ok. Now, what you do is let a 1 be an element of this subsequence ok, so choose a 1 to be any element of this subsequence.

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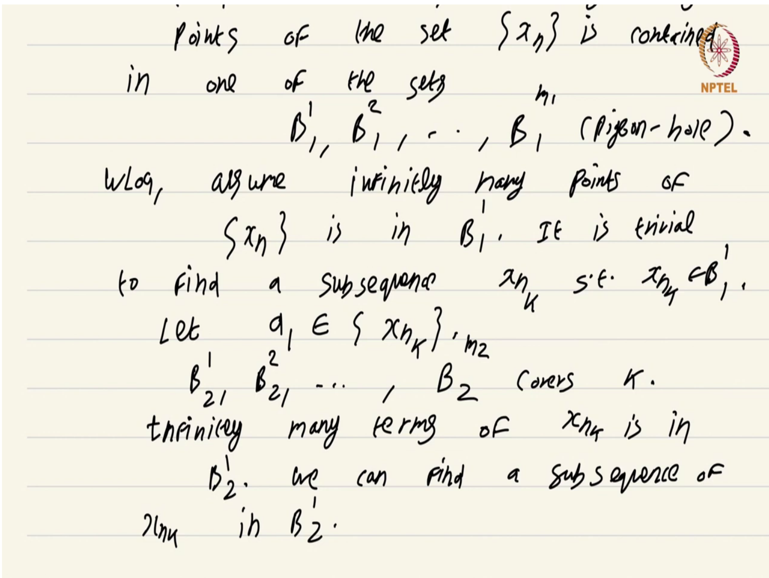

Now, what you do is that now is the tricky point of the proof, what you do is note that B 1 2, B 2 2 dot dot dot B m 2 2, I hope I got the notation right, yes, B m 2 2 covers K again right, covers K again. This just means that infinitely many terms, infinitely many terms of x n K is in B 1 2.

Again it will be there in one of these B I mean the infinitely many terms of x n K will either be in B 1 2 or B 2 2 or B m 2, I am just renaming these balls. And assuming that it is in B 1 2. There is nothing really deep happening here ok. Again we can find we can find we can find a subsequence of x n K, x n K in K, x n K in B 1 2 ok.

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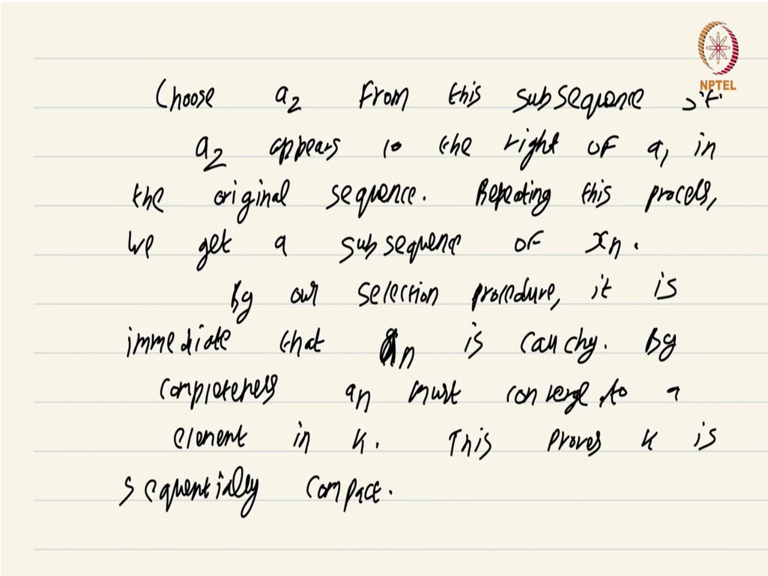

Now, what you do is choose a 2 from this subsequence from this subsequence such that a 2 appears to the right of a 1 to the right of a 1 in the original sequence in the original sequence fine. Now, repeating this process, repeating this process, repeating this process, we get a subsequence of x n. And by our selection procedure, by our selection procedure, it is immediate it is immediate that it is immediate that this sequence a n that we have chosen is Cauchy.

Now, by completeness a n must converge. Remember a n was a subsequence of the original sequence, we have shown that it converges and it must converge to an element in case simply because K is complete proving this proves K is sequentially compact sequentially compact. So, bit of work is required for this part. You have to consecutively choose these terms a n in such a way that it is a Cauchy sequence right.

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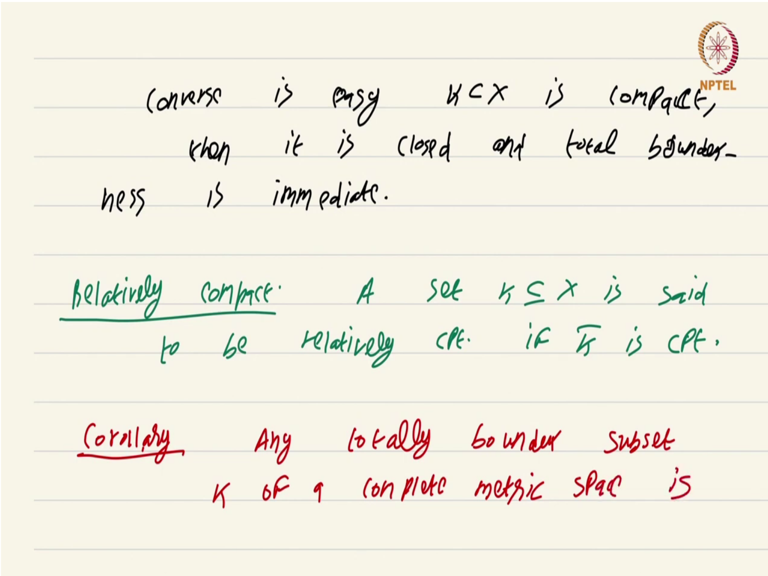

The converse on the other hand is quite easy. Converse is easy; converse is easy. Suppose K subset of X is compact, it is compact, then it is certainly closed. It is closed and total boundedness is easy.

You just use the fact that any cover of K with balls of radius epsilon will have a finite sub cover, and total boundedness is immediate. So, this concludes the proof of the Heine-Borel theorem for metric spaces. Again I urge you to show that I mean to come up with many examples of non totally bounded sets.

So, let me conclude by an with giving a notion that is very useful, this is called the notion of relatively compact. A set K subset of X is said to be relatively compact if K closure is compact ok. And we have an immediate corollary, we have an immediate corollary of

Heine-Borel theorem, it says any totally bounded set totally bounded subset K of a complete metric space complete metric space is relatively compact.

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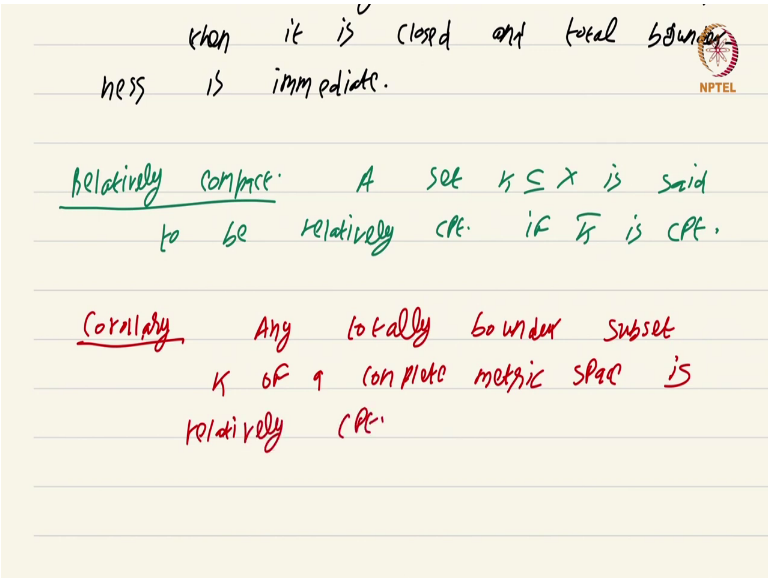

And the proof is rather easy or all corollaries are usually very easy.

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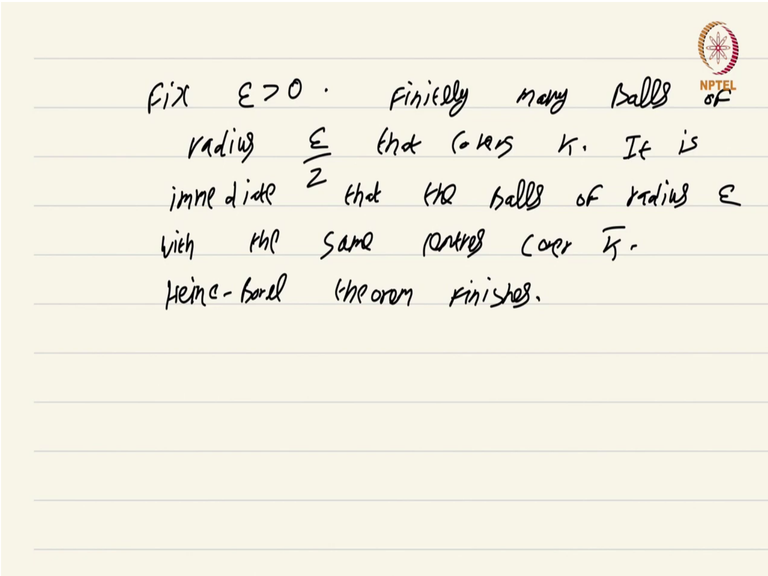

So, fix epsilon greater than 0. By the total boundedness of K, we can find finitely many balls finitely many balls of radius epsilon by 2 that covers K, that covers K. This is just the total boundedness. It is immediate, it is immediate that immediate that the same that the balls of radius epsilon with the same centers with the same centers cover K closure.

So, this statement might seem a bit cryptic. I do not want to belabor what is essentially an easy point. What I want to say is you have finitely many balls of radius epsilon by 2 that covers K. Just take these balls and delete the divided by 2. Just take the very same balls, but double the radius the same centers; those will have to cover K closer that is a rather easy thing to prove.

Now, the Heine-Borel theorem finishes the proof the Heine-Borel theorem finishes the proof. So, K closure will be closed and totally bounded and therefore, will be compact. So, this concludes the proof of the Heine-Borel theorem. As I have remarked several times, please do come up with several examples of non totally bounded sets.

This is a, this is a course on Real Analysis. And you have just watched the video on the Heine-Borel theorem.