Real Analysis II
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Lecture - 6.2
Open Covers and Compactness

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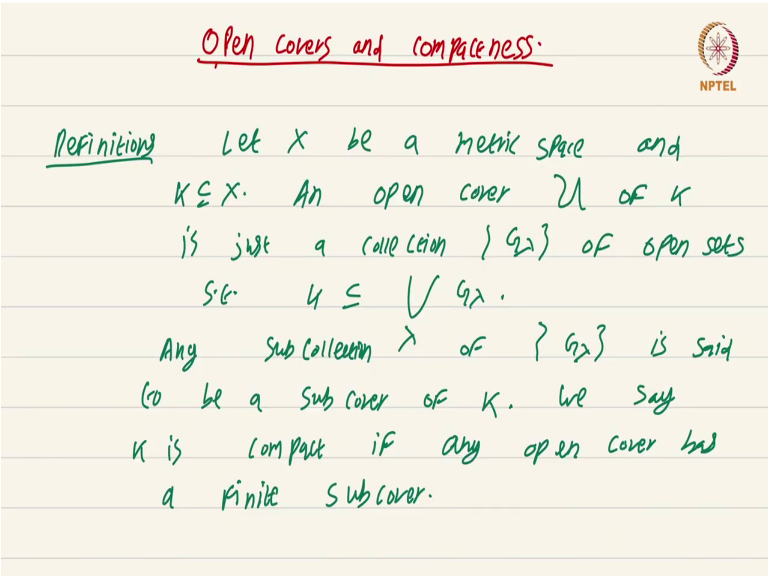

We are now going to characterize compactness in terms of open covers. The methodology of proof is exactly the same for the same result we showed for the real numbers. So, if at any point of time things seem to be going a bit abstract and complicated just look back through the proof in the real numbers to get a concrete feeling of what is happening. So, let us begin with the definition. And these are all rather straightforward definitions because we have already seen it once.

Definitions; let X be a metric space, and K subset of X. An open cover u of K is just a collection G lambda of open sets, I do not care what the indexing set lambda is – it could be finite, countable, uncountable, it could be anything; it is just a collection G lambda of open set such that K is a subset of union of G lambda. I just take the union over the indexing set of all these sets ok. Now, this is the definition of an open cover.

Any sub collection sub of G lambda is said to be a sub cover, is said to be a sub cover of K ok, so far so good. We say K is compact; K is compact if any open cover has a finite sub cover. So, you need to find a sub collection of the cover comprising finitely many sets whose union is also K. So, this is the definition of compactness; nothing new here it is exactly the same. Now, I am going to show that sequential compactness is same as compactness.

And again we proceed via an intermediary concept called the Lebesgue number. Now, here is a place where I should make a point way back when we studied topology of real numbers and studied all this in real numbers, I had defined a Lebesgue number in a particular way that was just to make the proof simpler in that scenario. Technically, what I defined as the Lebesgue number, it was not the correct definition.

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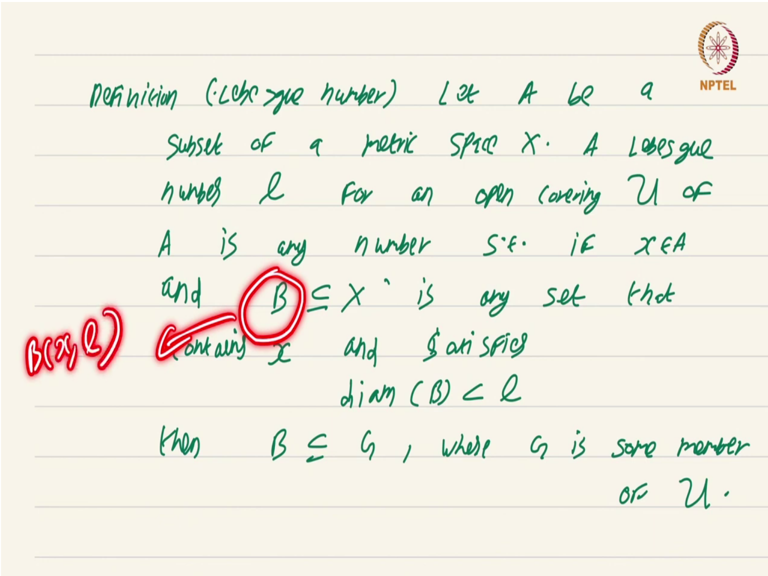

So, please note what I am going to define now is the correct definition of the Lebesgue number ok. So, this is the correct definition of the Lebesgue number that you can now delete your earlier definition from your brain and empty the recycle bin also. So, that this new fresh definition is what stays in your mind till you I mean till you retire from active mathematical life.

Let A be a subset of a metric space X. A Lebesgue number; a Lebesgue number let us just call it l for lack of creativity, a Lebesgue number l for an open covering u of A is any number, note this is not unique is any number such that if x is in A then B of x l is subset of G for some G in u ok.

So, what this says is that yeah. So, sorry about that I said that I am going to change the definition, and I gave the same definition again sorry. If x is in A, and B subset of X is any set not just a ball or anything, any set that contains x; that contains x, and is and satisfies diameter of B is strictly less than l, then B is subset of G where G is some member of u ok.

So, it was rather fortunate that I began with the wrong definition that was not intentional I promise you, it was complete accident. The earlier definition in the place of this B what I have called B, we had B x, l ok that is incidentally of diameter 2 l and not just l in general that is of diameter 2 l. It is not always true for weird metric spaces the diameter of that set could be 0.

What this new definition is saying is that you do not need to just consider balls, you just take any set B with the property that it contains some point x ok. It need not it could contain some point x in A, it does not matter what it is. And if its diameter is less than l, then it must automatically belong to some member of the open cover ok. So, the definition is same in spirit, but different in the details ok.

Now, I am not going to bother relating this definition with the earlier definition. There is an obvious relationship one notion of Lebesgue number will also be the other, but the other way around will not be true. I am not going to bother with all that neither should you. But let me just make the remark that qualitatively what we did earlier and what we are going to do now are not that different ok. So, let us now prove a sequence of results that finally get what we want.

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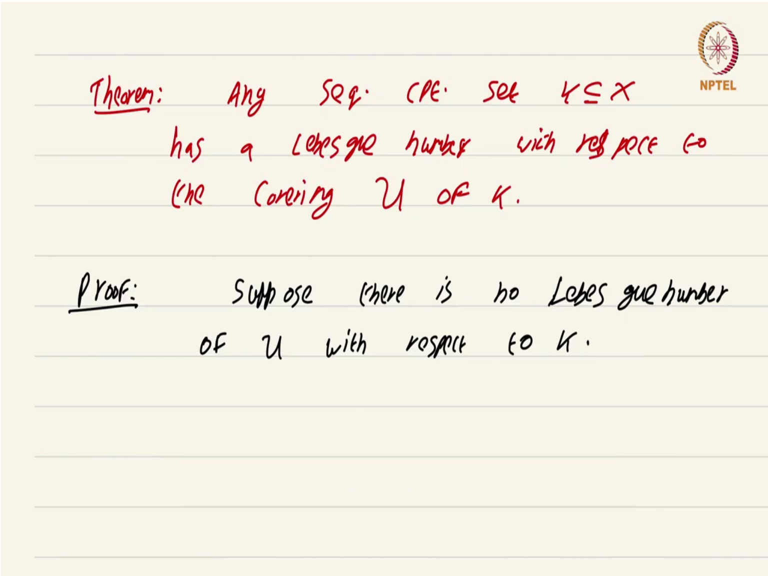

So, first result. Theorem, let me dignify it and call it a theorem any sequentially compact set K subset of X has a Lebesgue number; has a Lebesgue number with respect to; with respect to the covering u of K. So, whatever covering of K you take, it will always admit a Lebesgue number provided the set you start out with sequentially compact.

Proof; so what we do is the following. Suppose, there is no Lebesgue number for u, there is no Lebesgue number of u with respect to a with respect to K. Now, to be completely consistent with terminology, there is any has a Lebesgue number with any compact set K in X.

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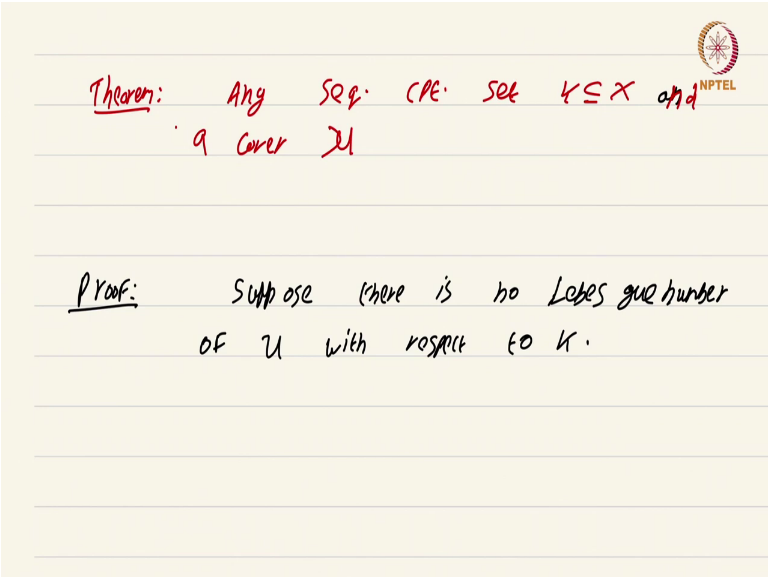

And a cover let me just rewrite this definition in mathematically precise terms. So, that I do not confuse you even further.

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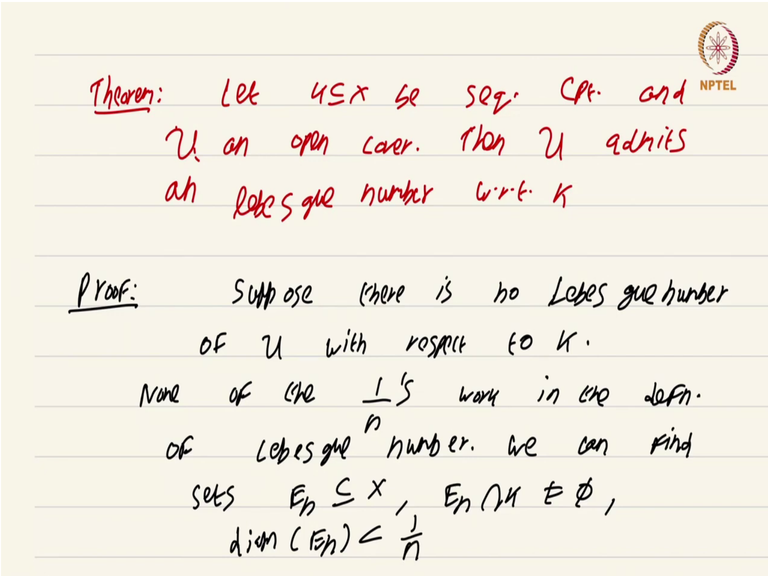

Let K subset of X be sequentially compact, and u an open cover ok. Then u admits an Lebesgue number with respect to A. So, I mean I just reversed what respects what in the statement that I gave earlier. This is the way I have formulated. There is nothing secret or writing it exactly the way it is, but this just is to have consistent terminology, so that we are precise ok.

So, suppose there is no Lebesgue number of u with respect to K, again I made a slight error here this should be K ok. Suppose, there is this covering u which does not have a Lebesgue number with respect to K. What does that mean? That just means that none of the 1 by n's work in the definition; in the definition of Lebesgue number right.

The only way by which this covering can fail to admit a Lebesgue number with respect to K is if none of the 1 by n's work which just means that we can find; we can find sets E n subset of

X, E n intersect K is non-empty, diameter of E n, diameter of E n is less than 1 by n; diameter of E n less than 1 by n such that E n is not fully contained in any member of u that is what the meaning of 1 by n not being a Lebesgue number means ok.

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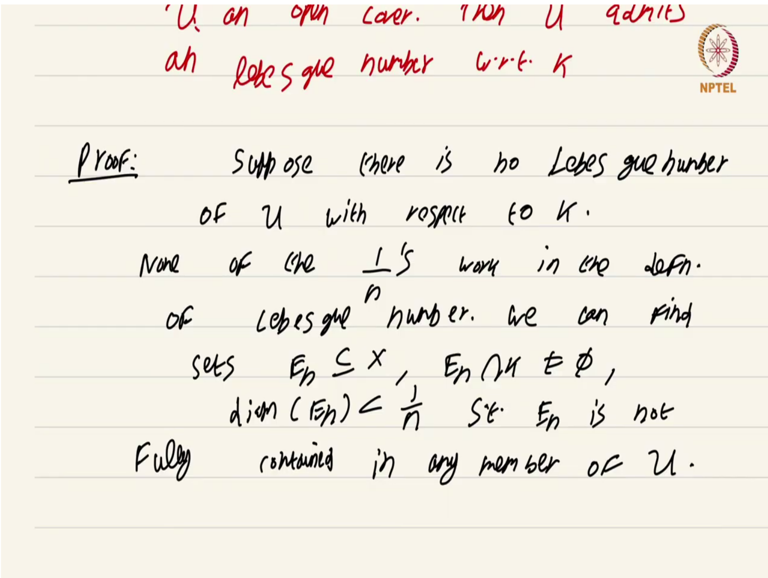

Now what you do is the following?

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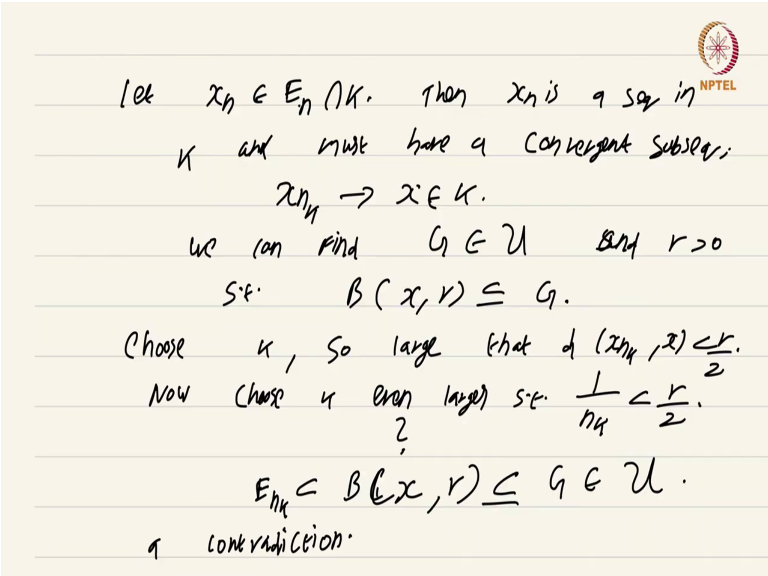

Let x n be some element in E n ok rather let x n be some element in E n intersect K. Let x n be an element in E n intersect K. Then x n is a sequence in K and must have a convergent subsequence right subsequence x n K converging to x, and this x must also be in K ok. Now, here is the key thing since x is in K, we can find; we can find G in this collection u and r greater than 0 such that B of x comma r is fully contained in G.

Why? Because u is an open covering of K, x is a point in K, that means, we can find some element G in this open cover which contains the point x. But G is open that means, we can find some radius such that B x, r is fully contained in G.

How does this help us? Well, choose K so large choose K so large that d x n K, comma x is less than r by 2. Of course, we can do this simply because x n K converges to x ok. Now, choose K even larger; choose K even larger such that 1 by n K is also less than r by 2.

So, just choose this K so large that both these conditions are satisfied the distance from x n K to x is less than r by 2, 1 by n K is also less than r by 2 ok. Then observe that E n K, E n K has to be contained in B x n K comma r. I want you to check why this is true its not really hard. And this is contained in G which is contained in u ok.

So, this is an easy check to check that E n K is actually contained in just wait a moment, I made a slight mistake E n K is contained in B x, r is contained in B x, r which is contained in G which is an element of u, a contradiction ok. So, this is not really hard to show that E n K is contained in B x, r. By hypothesis B x, r is contained in G I mean is a subset of G and that is an element of u.

So, this shows that any sequentially compact set in a metric space must have a Lebesgue number. So, we now have all the tools ready at our disposal to prove equivalence.

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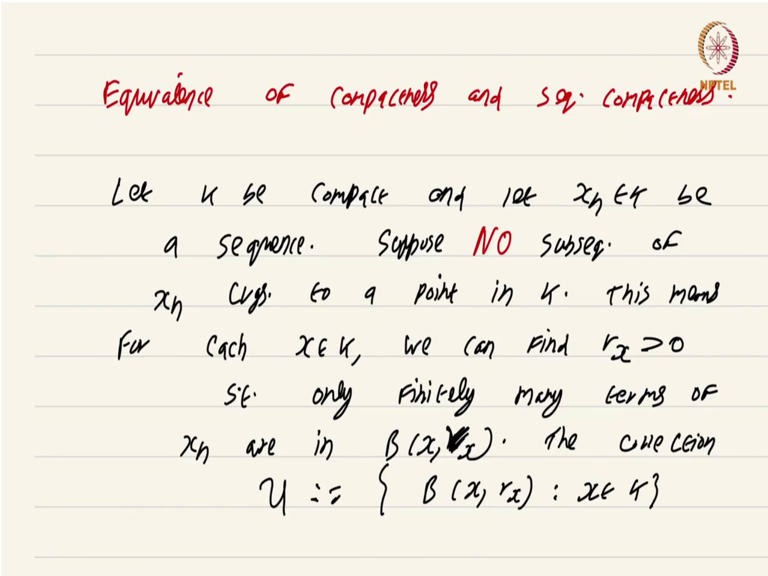

Equivalence of compactness and sequential compactness, all the tools are ready. So, we can show that both notions are equivalent. So, let us begin the proof right away I am not going to dignify such an easy statement with a rigorous mathematical statement. I am just going to show that a set K is compact if and only if it is sequentially compact and vice versa.

So, let us start with let K be compact and let x n; let x n in K be a sequence. The goal is to show that x n admits some subsequence that converges to a point in K that is the definition of sequential compactness. Suppose, no subsequence of x n converges to a point in K ok.

Now, I am going to make a statement that might seem a bit weird, but it is just a matter of sitting down and really understanding what suppose no subsequence of x n converges to a point in K means.

So, I claim this means for each x in K, we can find; we can find r x greater than 0 such that only finitely many terms of x n are in B x, r x ok. This is just capturing the fact that the sequence cannot get arbitrarily closed to any element of K because if it does then we will be able to extract a subsequence that converges to that particular point ok.

So, I am just saying that no matter what point you take you can find an r x – a radius positive radius such that only finitely many terms of the sequence are there in this ball of radius r x centered at X ok. Now, because this is true for every single point, the collection of such balls the collection u which is by definition B of x r x as x runs through K is an open cover; is an open cover of K ok.

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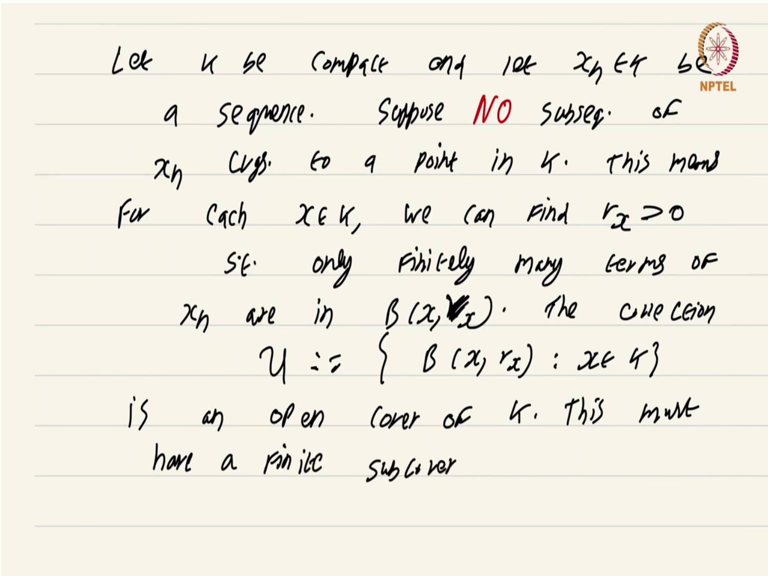

Now, this must have a finite sub cover ok. I am going to simplify notation a bit.

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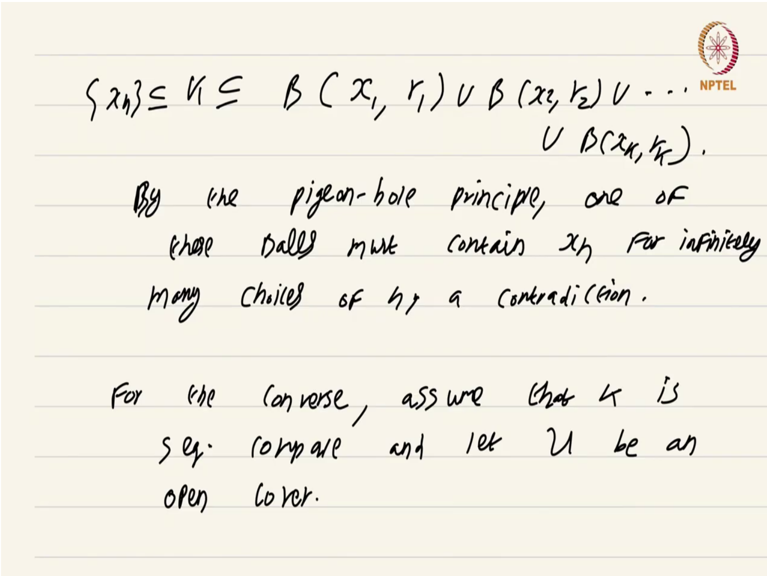

So, there is this finite sub cover. So, I can write K as a subset of B x 1 r 1 union B x 2 r 2 union dot dot dot B x K, r k. So, this compact set K will be contained in this finite unit. I have just simplified a notation bit technically I should write B x 1 r x 1, I have just simplified it with the subscript 1 as there is no scope for confusion.

Now, this sequence x n, this set is also a subset of this which means by the pigeonhole principle, by the pigeonhole principle one of these balls; one of these balls must contain x n for infinitely many choices of n ok. So, again let me just go back here. I should clarify one bit here I have written for each x in K, we can find r x greater than 0 such that only finitely many terms of x n. So, there is a scope for confusion only for I will just rewrite this statement.

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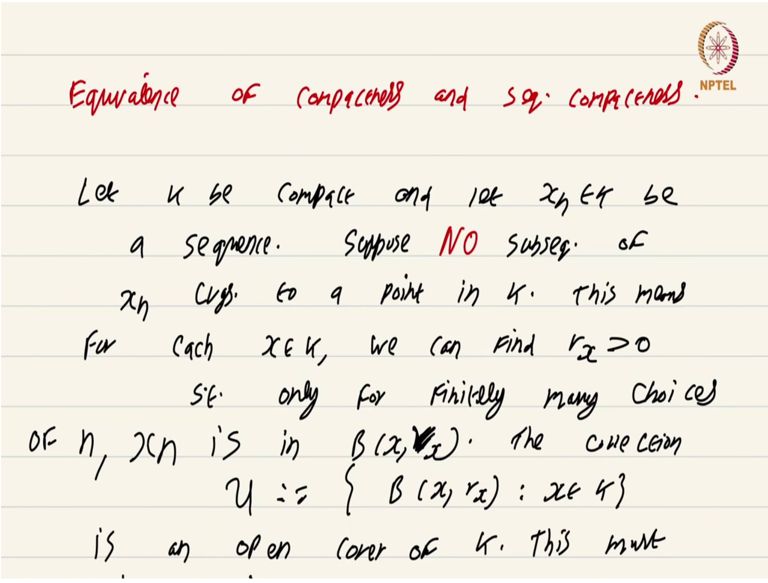

Only for finitely many choices of n; many choices of n, x n is in B x, r x. The way I have written it previously there is a slight ambiguity, do I mean that there are infinitely many distinct points or do I mean that there are infinitely many ns for which x n is in B x, r x. So, I have clarified it now. So, it should be clear to you that we have obtained the direct opposite of our hypothesis.

We have found by the pigeonhole principle that one of these balls must contain x n for infinitely many choices of n a contradiction. So, this concludes the proof that compactness implies sequential compactness. For the converse, for the converse, for the converse, assume that K is sequentially compact; assume that K is sequentially compact. And our goal is to show that it is compact.

And so we will take. And let u be an open cover; be an open cover our goal is to produce a finite sub cover. So, what we are going to do is the following. We already know that sequentially compact sets admit Lebesgue numbers with respect to whatever covers or I have already forgotten which way it goes.

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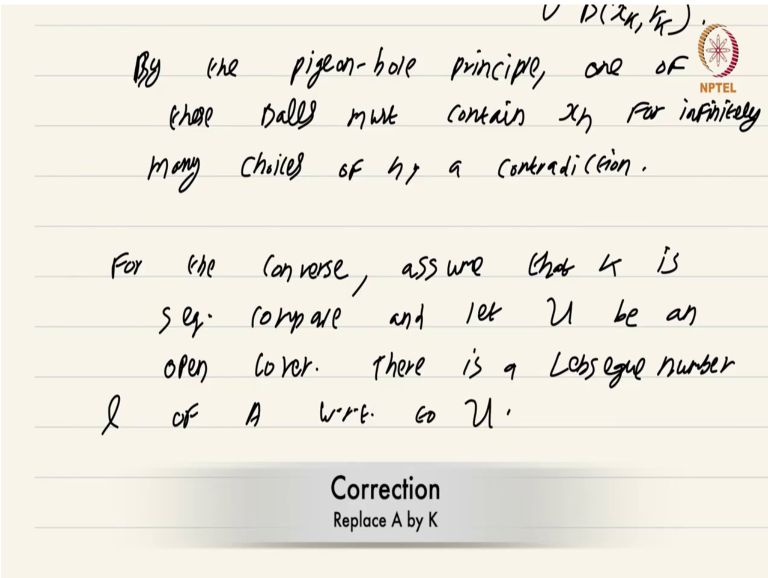

So, I think this is the correct way there is a Lebesgue number l of A with respect to u. I can find a Lebesgue number for this open covering.

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$\cup B(x_k, r_k)$.

By the pigeon-hole principle, one of these balls must contain x_n for infinitely many choices of n, a contradiction.

For the converse, assume that K is seq. compact and let $\mathcal{U}$ be an open cover. There is a Lebsegue number l of A w.r.t. to $\mathcal{U}$. If $x \in K$,

$$B\left(x, \frac{l}{2}\right) \subseteq U_1 \in \mathcal{U}.$$

Now, this just means that if x is in K, if x is in K, then B x comma l by 2; comma l by 2, this is going to be contained in some element u, some element u which is an element of u right. This is just the definition I will just call this u 1 ok. I will just call this u 1. So, what you do is the following. What you do is I just a slight change.

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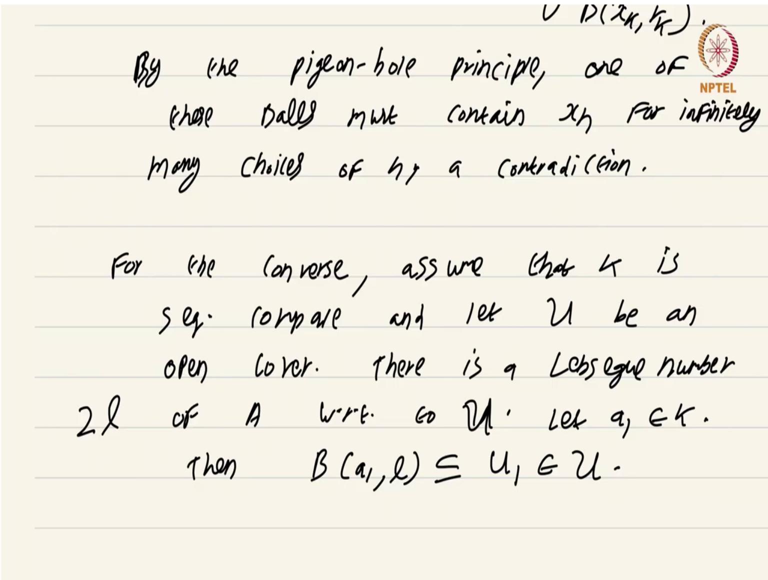

What I am going to do is, what I am going to do is let a 1 be an element of K, let a 1 be an element of K. If you do not mind, let me make a slight change that will make my life and your life easier let me put a 2 here, I have just made this 2 l, let 2 l be a Lebesgue number of A. Note and the reason why I am doing this any number lesser than a Lebesgue number is also a Lebesgue number.

So, let a 1 be in K, then by what I just said B a 1, l is going to be a subset of u 1 which is going to be an element of u ok. There is going to be some u 1 in this open cover u that contains the entire ball of radius l centered at a 1.

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Suppose $K \setminus B(a_1, l)$ is non-empty.

Choose $a_2 \in K \setminus B(a_1, l)$ and $U_2 \in \mathcal{U}$

$$B(a_2, l) \subseteq U_2 \in \mathcal{U}.$$

Repeat this. If the process does not terminate, we can find a sequence

$a_1, a_2, \ldots \in K.$

$$d(a_i, a_j) \geq l \quad \forall\, i \neq j.$$

No subsequence of a_n can possibly cvg!.

Now, suppose K minus B a 1, l is non-empty. So, here is the idea I want to extract a finite sub cover of this cover u in some manner. What I am doing is, I am going to do the naivest thing possible I am going to take a given point a 1 and K, then I am I know that by the fact that this cover as a Lebesgue number. I will be able to find a u 1 which contains the ball of radius l centered at u 1.

If it so happens that just B a 1, l encapsulates the whole of K, that means, u 1 also encapsulates the whole of K which just means that I can just stop this procedure. But if K minus B a 1, l is non-empty, we can continue. Choose a 2 in K minus B a 1, l, and u 2 in u such that B a 2, l is a subset of u 2 is a subset of u 2 which is an element of u, exact same thing I am doing.

Now, if at some finite point this process I am going to repeat this, if at some point finite point this sequence if; this sequence if at some finite point I am unable to choose any more points or in other words if those finitely many balls B a 1, l, B a 2, l dot dot dot B a K, l if they happen to cover K, then I cannot proceed any further. I have found the required finite sub cover that is just u 1, u 2 dot dot dot u k.

On the other hand, if it happens infinitely often, so repeat this if the process does not terminate, if the process does not terminate we can find a sequence; we can find a sequence a 1, a 2 dot dot dot in K right. And we can find an infinite sequence. Not only can we find an infinite sequence, we can find an infinite sequence with a very peculiar property that d a i, a j is greater than or equal to l for all i not equal to j.

If you take two points in this sequence there will be at least this Lebesgue number apart, and that is just by the way we constructed. For instance, just to see this we chose a 1 and we chose a 2 coming from K minus the ball of radius l centered at a 1. Therefore, the distance between a i and a j must be at least l ok. Now, what this shows is that no subsequence of a n can possibly converge; can possibly converge right.

This is simply because any term in this sequence is at least l distant away from any other term. So, no this same thing will be true for any subsequence. And this subsequence therefore cannot be Cauchy ok which just means that it cannot be convergent ok.

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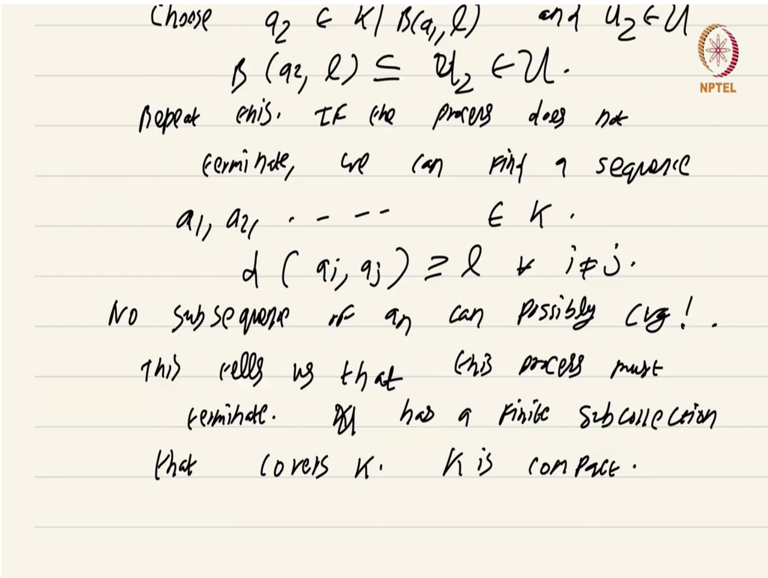

Now, what does this tell us? This tells us this tells us; this tells us that; tells us that this process must terminate, this process must terminate which is just a fancy way of saying, fancy way of saying u has a finite sub collection; finite sub collection that covers K right. So, again the proof is not really hard. It just uses some basic ideas about Lebesgue numbers. And we will be able to conclude that K is compact.

So, K is compact we have now shown both directions that K is compact if it is sequential compact; and K is sequentially compact if it is compact. So, we have characterized compact sets in several ways. In the next video, we will try to get a version of the Heine-Borel theorem that is true for all metric spaces.

As we have already seen the Heine-Borel theorem is not true in all metric spaces, we were able to show it for r n, but we have still not explored what is the correct version of

Heine-Borel theorem for metric spaces. For that we need something called total boundedness which is the topic of the next video.

This is a course on Real Analysis, and you have just watched the video on Open Covers and Compactness.