Real Analysis II
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Lecture - 6.1
Bolzano--Weierstrass Property

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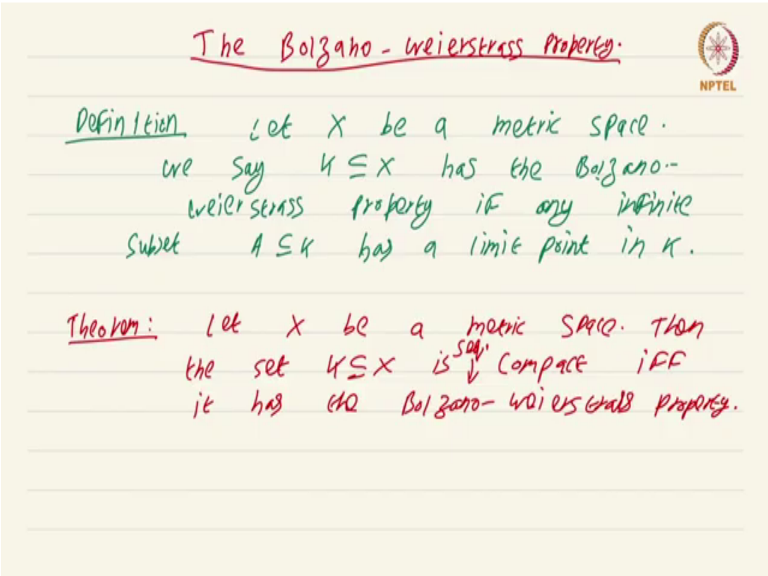

We are now going to characterize the notion of compactness in several ways. In this short video we warm up by proving that compactness is equivalent to having the Bolzano-Weierstrass property. What is the Bolzano-Weierstrass property? Well, I am going to define that now.

Definition: let X be a metric space be a metric space. We say K subset of X has the Bolzano-Weierstrass property if any infinite subset A of K has a limit point in K. So, this set

K in the metric space X is said to be Bolza said to have the Bolzano-Weierstrass property if whatever infinite subset of K you choose it will definitely have a limit point in K.

Notice that you need to consider only infinite subsets in this. Furthermore it only says that there is at least one limit point which is in K. It does not tell you that every limit point of A is actually going to be in K or any such thing, no such assumption is made. Just one limit point has to be there in K. Now, one more comment as you could have guessed from the definition any finite set is automatically going to have the Bolzano-Weierstrass property.

I urge you to think of more examples in the setting of the real numbers and in r n, which are the most important metric spaces for r purposes. Anyway what I am going to prove now is the fact that the fact that compactness and having the Bolzano-Weierstrass property are equivalent.

Let X be a metric space be a metric space then the set K subset of X is compact if and only if it has the Bolzano-Weierstrass property. So, what this says is checking whether a set is compact. So, again I keep forgetting the adjective, this is sequentially compact ok.

They are all the same. I am not just being lazy and forgetting. It is just that subconsciously I have already identified all the various notions for metric spaces. You will also be in this position a couple of videos down the line till then we have to be a bit careful and write sequential compact. So, set is sequentially compact if and only if it has the Bolzano-Weierstrass property and the proof is rather short. So, let us begin the proof.

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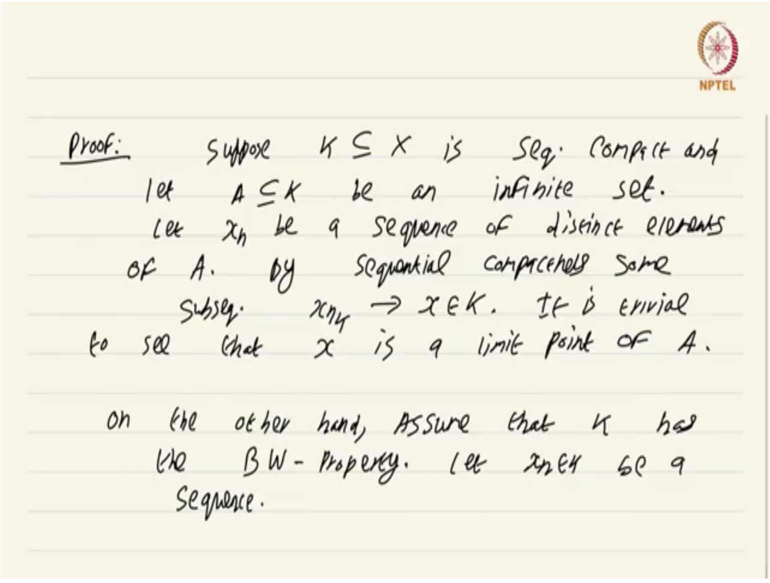

Proof: now, what I am going to do is I am first going to show that any sequentially compact set must necessarily have the Bolzano-Weierstrass property. So, suppose K subset of X is sequentially compact and let A subset of K be an infinite set be an infinite set. Now, I have to show that some limit point of the set A is contained in the set K. So, what I do is let x n be a sequence of distinct elements of A; sequence of distinct elements of A, ok.

Now, why does such a sequence exist? Because we are assuming that A is an infinite set, we can definitely find an infinite sequence of distinct points. By sequential compactness by sequential compactness some subsequence x n K converges to x and this x should be in K that is the definition of sequential compactness.

Now, it is trivial to see that it is trivial to see that x is a limit point of A, ok. So, one part this part is rather just apply the definition of sequential compactness and you get what you want almost immediately. Now, on the other hand; on the other hand assume that K has the Bolzano-Weierstrass property. I am getting tired of remembering the correct spellings of Bolzano and weierstrass.

So, I am just going to write BW property for the sake of not making a typographical error in the name ok. Now, let x n in K be a sequence. Our goal is to exhibit a convergent subsequence of x n and this subsequence should converge to a point in the set K that is our goal.

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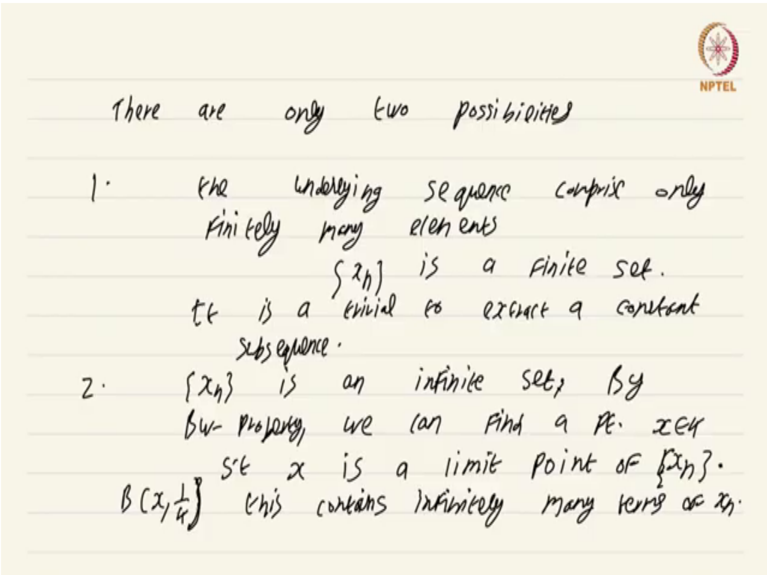

Now, there are only two possibilities there are only two possibilities there are only two possibilities. The first property says the underlying sequence the underlying sequence comprises only finitely many elements finitely many elements. What this essentially means is that this x n set is a finite set. You put all the terms of the sequence inside a set and you get a finite set that is one possibility. In this case it is trivial to extract a constant subsequence; it is trivial to extract a constant subsequence.

So, if finitely many terms just keep repeating at least one term should keep repeating infinitely many times and that will give you thus constant subsequence which is obviously, converging to a point in K. So, that is one possibility the second possibility is this set x n is an infinite set is an infinite set ok.

And x n being an infinite set immediately by Bolzano-Weierstrass property Bolzano-Weierstrass property we can find; we can find a point x in K such that x is a limit point x is a limit point of this set x n. Note this is a limit point of the set x n not I repeat this is not a limit of the sequence x n or any such thing ok.

Now, from this it is rather easy to extract a subsequence. I am going to leave that to you as a straightforward exercise, but let me just give you a hint. If you look at this B x 1 by n, this ball of radius 1 by n this contains infinitely many contains infinitely many terms. So, let me not use K, let me not use n let me use K instead.

Take this ball of radius 1 by K, this contains infinitely many terms of x n ok. The sequence x n infinitely many terms of the sequence x n has to be there in this ball. In fact, that is the characterizing property of a limit point. So, from this it should be rather straightforward to extract the required subsequence that converges to X showing that any sequence has a convergent subsequence in K. Hence, K is sequentially compact.

Now, in the next video I am going to formulate a compactness in terms of open covers. So, stay tuned for that. This is a course on Real Analysis and you have just watched the video on the Bolzano-Weierstrass property.