Real Analysis II
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Lecture - 5.2
Compactness

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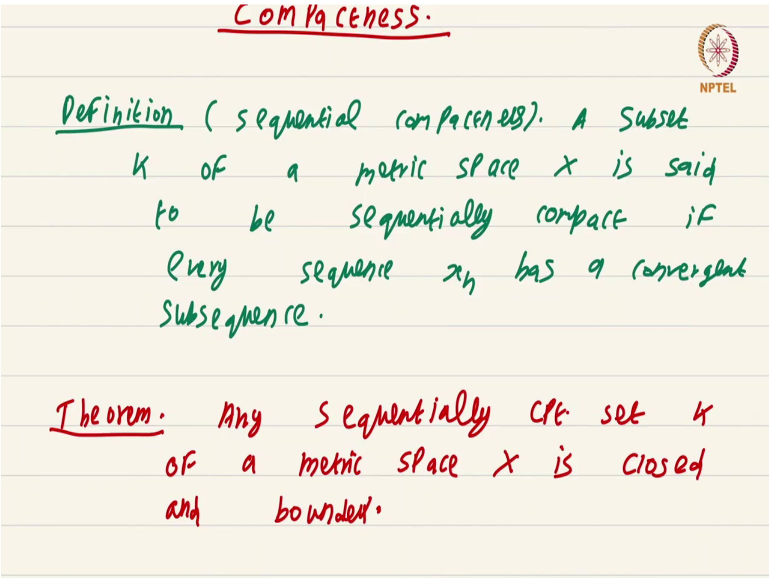

We now study what is no doubt the most important topological concept, the notion of Compactness. We have already spent a lot of time on this topic when we studied topology in the real line, much of that will transfer quite easily to the more general setting of metric spaces.

However, there is one major caveat; the Heine Borel theorem is not entirely true in the case of metric spaces. You will understand the importance of compactness when you study more advanced courses like functional analysis or Riemannian geometry.

For the time being, just take it for granted that what you are about to study is not only extremely interesting, but extremely important as well. We begin with the definition and the definition should be very very familiar to you; this is the definition of sequential compactness of sequential compactness, it says the following.

A metric space X or let me take a slightly more general definition, a subset K a subset K of a metric space X is said to be sequentially compact if every sequence x n every sequence x n has a convergent subsequence.

So, the definition is exactly the same as what we saw for the real numbers. So, immediately we are going to prove a theorem, which is sort of one direction of the Heine Borel theorem which is true; the other direction which is famous is not true, you need to add some more conditions to make the other direction true.

So, this says the following; any sequentially compact set any sequentially compact set K of a metric space of a metric space X is closed and bounded. So, this direction even though I have dignified it by calling it a theorem is really just a basic fact that you can prove in a few lines, let us see the proof.

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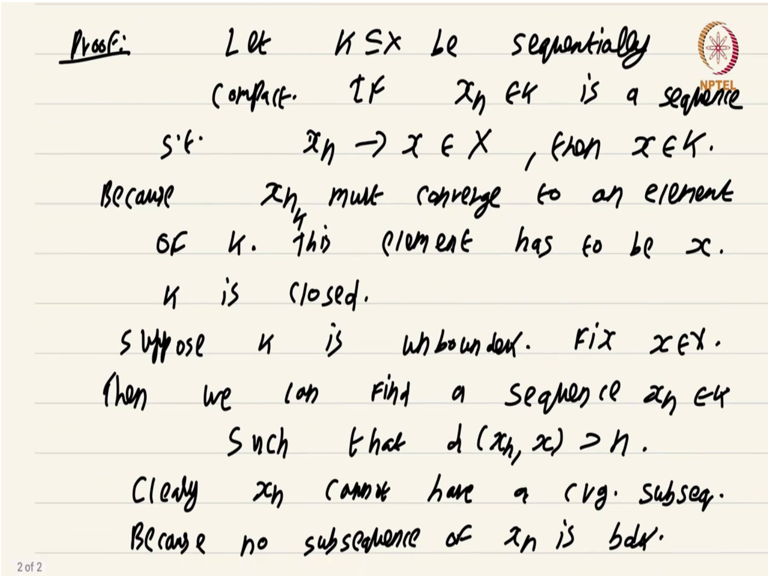

Let K subset of X be compact or sequentially compact, let it be sequentially compact. So, let me just, yeah we are in the metric space X. We have to show that X is closed and bounded. So, first of all if x n in K is a sequence is a sequence, such that x n converges to x in X, then x must necessarily be in K. Why is this? Because some subsequence x n K must converge to an element of x, this is the definition of sequential compactness, ok.

Because K is sequentially compact and x n is a sequence in K, we must have some subsequence x n K that converges to an element of K; this element has got to be x itself, this element has to be x, ok. This shows that K is closed. What we have shown as any adherent point of K is an element of K, ok.

To show boundedness, suppose K is unbounded, suppose K is unbounded; what we are going to do is, we are going to construct a sequence in K that cannot possibly have a convergent subsequence, which will violate compactness, therefore K must be bounded. Suppose K is unbounded, fix x in X, it does not matter what point this is; then we can find we can find a sequence x n in K, such that the distance of x n to x is greater than n.

So, this actually requires a few moments of thought or a few lines of proof; if this were not true, if it is the case that every element y in K is some finite distance away from the point x, then if you call that distance let us say capital N, the ball of radius capital N centered at the point x will in fact contain the set K contradicting the fact that K is bounded.

So, think about this for a few moments or sit down and write a few lines to make this part rigorous; it is rather easy, so I am skipping it. Clearly x n cannot have a convergent subsequence, because no subsequence is bounded, no subsequence of x n is bounded; that is the way this sequence x n has been constructed.

This shows that it cannot be the case that K is unbounded, so K must be bounded, ok. So, this shows that K must be bounded; that means K is both closed and bounded which is what we wanted to prove. Now, you might wonder why the converse is not true and the only way to really see that the converse of a statement is not true is to actually see an example, where it is not true.

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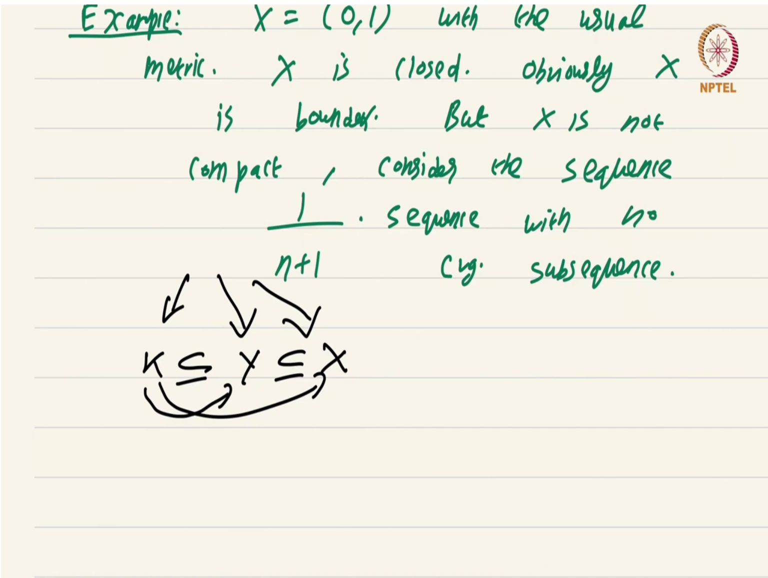

So, let us take the example, what we do is the following; just consider our X metric space to be 0, 1 with the usual with the usual distance or metric, usual metric the Euclidean metric on r, ok.

Now, observe this funny thing that happens; we are treating X itself as a metric space in its own right, so X is closed. Now, this is a silly thing that happens; if you are finding this a bit weird, go back a few videos where I talk about relative I mean subspaces and how openness and closeness are with respect to a metric space, it is not some notion that is intrinsic, ok.

So, X is closed, obviously X is bounded, obviously X is bounded ok; but X is not compact, X is not compact, just consider the sequence 1 by n plus 1. This is a sequence of elements in 0, 1 that converges to the point 0 actually; but 0 is not there in the metric space. So, this is a sequence with no convergent subsequence, ok.

So, closed and bounded sets need not be compact. So, this example makes us pause and think about compactness a bit more; because I made comments about openness and closeness being with respect to a metric space. So, the following question is natural.

So, suppose you have a metric space X and you take a subspace Y and then you take a subset K. Now, we know that K being open or closed depends on whether you treat the parent metric space as Y or X; depending on whether you are treating X as the metric space under consideration or Y as the metric space under consideration, whether K is open or closed will change.

Is the same thing; does the same thing happen for compactness? That means, suppose K is compact when treating Y as the matrix space; does it mean that K is compact or not compact, it could it be not compact as when you treat it as a subset of X and vice versa.

Suppose K treated as a subset of X is compact; then what does it imply for K considered as a subset of Y? Unlike the case of open sets and closed sets, surprisingly compactness does not depend on which metric space you consider; whether you consider Y or X, it really does not matter and that is the content of this next proposition.

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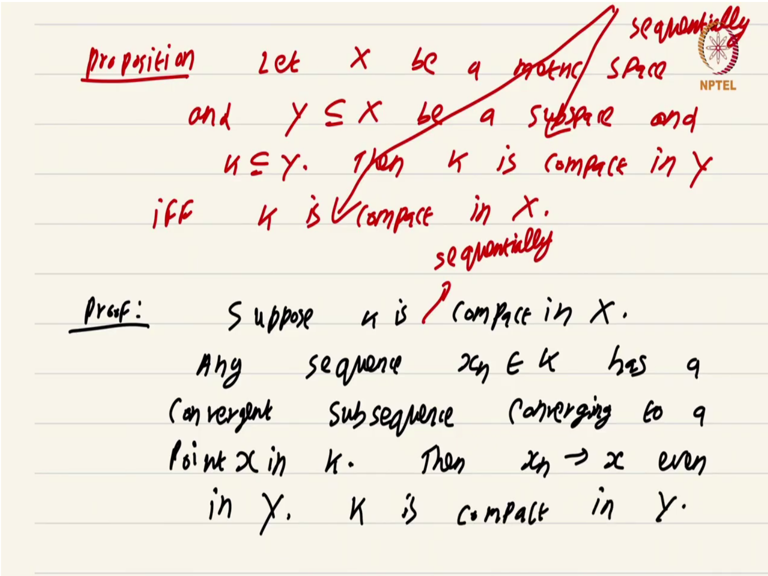

Proposition, let X be a metric space and Y subset of X be a subspace and K is a subset of Y; then K is compact in Y, if and only if K is compact in X, ok. So, henceforth we need not actually say compact in X, compact in Y, it makes no difference; it is going to be compact irrespective of which metric space you are considering K as a subset of.

Now, this is one of those scenarios where the proof is significantly harder than, I mean significantly easier then what you might first think, ok. Suppose K is compact in X, K is compact in X; what does that mean? Ok. To be hundred percent precise, I am just going to; because this will become important a bit later, just add the adjective sequentially, right. We have not yet defined compactness without any adjective yet.

So, suppose K is sequentially compact in X ok; that means any sequence x n in K has a convergent subsequence converging to a point in K, to a point in K, ok. Now, observe that the metric on Y and the metric on X are exactly the same; you just restrict the metric on X to the metric on Y. So, call this point x in k, then x n converges to x even in y; because the metrics are exactly the same. So, I here I should write capital Y, ok. So, that means K is compact in Y, ok.

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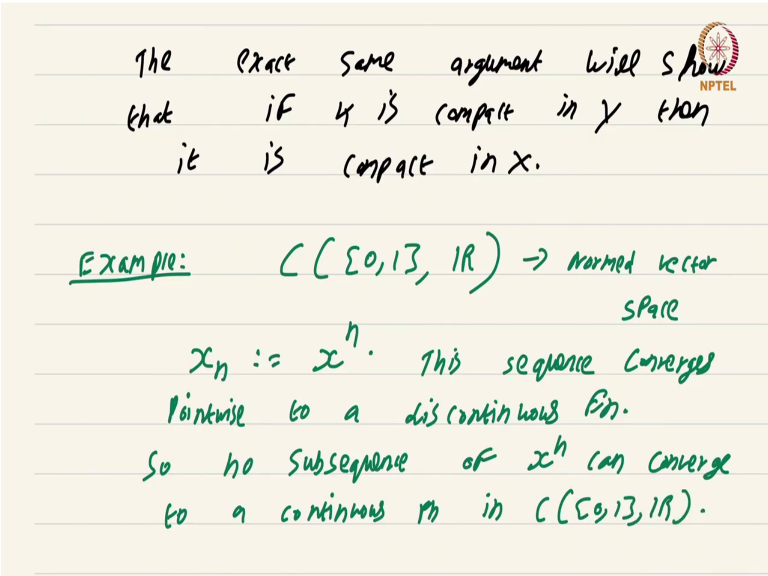

The exact same argument, the exact same argument will show that if K is compact in Y compact in Y, then it is compact in X. So, compactness is not a relative notion, it does not depend on which metric space you are considering, which I mean treat which metric space you are treating K as a subset of, ok.

Now, what we want to do is, I want to give a non trivial example of a closed and bounded set which is not compact. We have already seen one example, but that was rather trivial, I want to give another example to reinforce what is happening and this is slightly non trivial.

So, the example is the following, consider the space of continuous functions from the closed interval 0, 1 to R ok; this is actually a normed vector space, in fact it is complete, we have even shown that it is complete. So, this is the space of continuous functions from close 0, 1 to R. Now, what I am going to do is, I am going to consider the sequence x n which is given by x power n; these are all obviously continuous functions on 0, 1, so this is a the this is a sequence in C 0, 1, R.

Now, this sequence converges pointwise to a discontinuous function. We have already seen this example when we studied uniform convergence and pointwise convergence and all that. So, this sequence x power n converges to a discontinuous function; it is going to be essentially 0 on close 0 open 1 and 1 when x equal to 1, we have already seen this.

So, no subsequence of x power n can converge to a continuous function in C 0, 1, R. The reason for this is, you will have to work out an exercise that I have given earlier; the metric on C 0, 1 captures a uniform convergence, that is a sequence of functions f n in C 0, 1, R converges to an element f in C 0, 1, R in the sup norm if and only if f n converges uniformly to f.

But here x power n converges pointwise to a discontinuous function, therefore any subsequence of x power n must also converge pointwise to a discontinuous function; but we already know that uniform limit of continuous functions must be continuous. So, putting all these remarks together, it follows that there is no way that some subsequence of x power n can converge in C 0, 1, R to an element x in C 0, 1, R, not to an element x, to a function f in C 0, 1, R, that is simply not possible.

So, this gives another non trivial example of a closed and bounded set that is not that is not compact.

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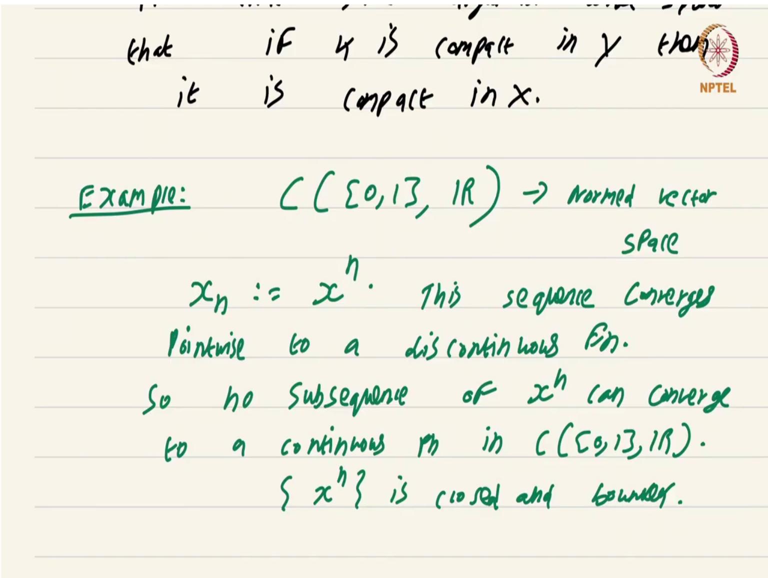

Now, you still have to show, you still have to show that this set x power n is closed and bounded for this example to work, right. This and it will be rather easy to show, this will be a bounded set; because you can compute the sup norm of all of this is going to be one, therefore this is certainly going to be a bounded set.

Why is it a closed set? That please think about it for a moment, it is rather easy; the argument that it is a close set is actually contained in this dialogue or rather monologue that I went through a couple of minutes ago. So, this gives a non trivial example.

Now, the next goal is to characterize the sequentially compact subsets of R n, ok.

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Theorem: Any closed subset of a sequentially cpt set K is sequentially compact.

Proof: Suppose $A \subseteq K$ is closed.
If $x_n \in A$ is a sequence then it has some subsequence $x_{n_k} \to x \in K$.
But x is an adherent point of A and therefore $x \in A$.

So, we will first prove a very simple theorem, which is going to be very useful in the future; theorem any closed subset of a sequentially compact set K is compact or rather sequentially compact, is sequentially compact. The proof of this is very easy.

So, suppose A subset of K is closed ok; here it is implicitly assume that when I say closed, I mean closed in the parent metric space, all of this is happening in a parent metric space X, ok. Now, if x n in A is a sequence is a sequence, then it has some subsequence x n K that converges to x in K. But x is an adherent point, is an adherent point of A and therefore, x is in A. This shows that any sequence in A has a subsequence that converges to an element in A showing that A is compact.

The other fact we need to characterize the compact, sequentially compact subsets of R n is the fact that a product of compact sets is compact.

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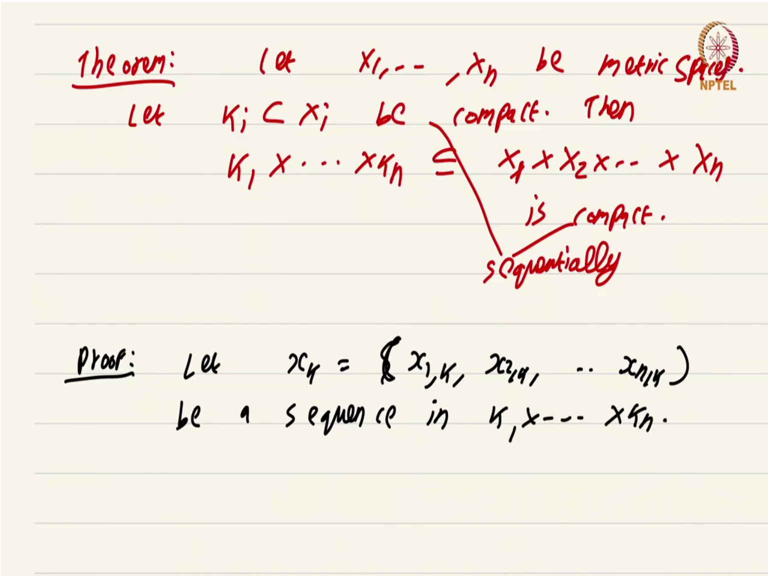

So, let X 1 to X n be metric spaces; let K i subset of X i be compact, then K 1 dot dot dot K n the Cartesian product, which is a subset of X 1, X 2 dot dot dot X n is compact or rather sequentially compact. Again I sometimes I will miss the adjective sequentially compact, it really does not matter; because ultimately when I define the notion of compactness, I will show that both notions are same. So, it really does not matter.

Now, before we begin the proof, you must get a little bit of uneasiness, because I have not told you what metric I put in the product X 1 cross X n; but recall from our discussion of equivalent metric spaces and product spaces, I said that we will not worry too much about what metric is there in the product. As long as it satisfies the characteristic property, that a sequence in the product converges if and only if each component converges, right.

So, let x K equal to x 1, K x 2, K dot dot dot x n, K be a sequence in K 1 dot dot dot K n, ok.

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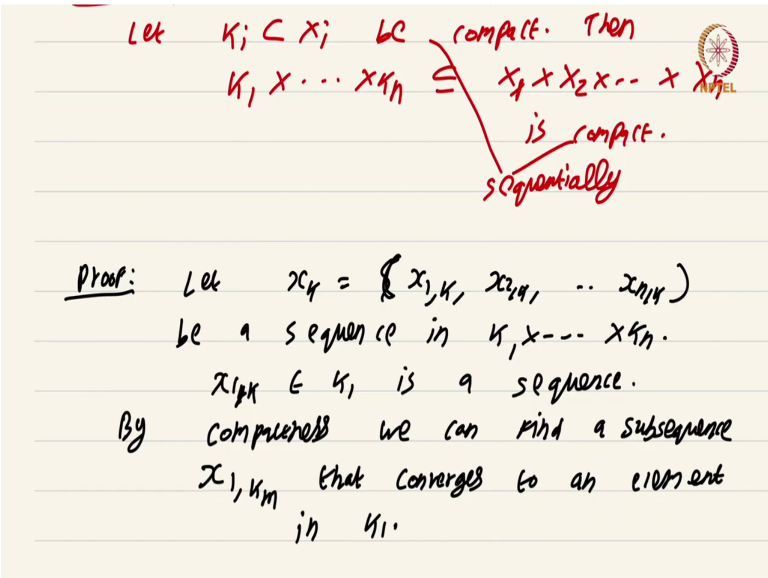

Now, x 1 K is therefore, in not in not just in x 1, it is there in K 1 is a sequence right; I am just extracting the first component.

By compactness, we can find we can find a subsequence. Let us just I need some notation for this; what I am going to do is, I am just going to call it x 1, m K. We can find a sub sequence x 1, m K that converges; rather I should just reverse it, it is K m, I can find a subsequence x 1, K m that converges to an element to an element in K 1.

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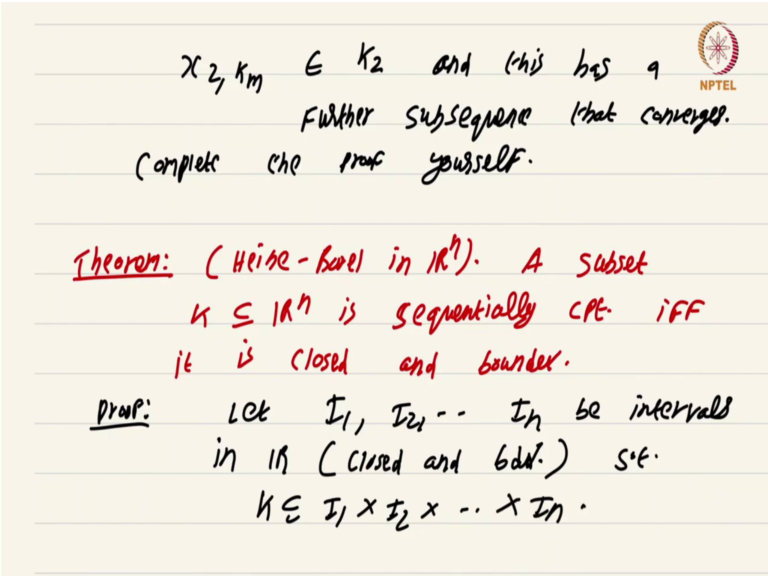

Now, what I am going to do is, I am going to consider the corresponding subsequence of x 2 K; that is I am going to consider x 2, K m which is a sequence in K 2 and this has a further subsequence, this has a further subsequence that converges, right.

Since the notation is going to become extremely complicated and well above my intellectual capability, I am not even going to bother writing down what is going to happen. What I am going to do is, I am going to repeat this argument for each coordinate factor successively taking subsequences; finally I will get a tuple x 1 to x n, such that you have a subsequence of x K that converges to this point.

So, I am going to just say, because of cover this, I am going to say complete the proof yourself, complete the proof yourself. So, once we have these two facts, it is very easy to find out what the compact sets in R n r; we have the theorem Heine Borel theorem in R n.

A subset K of R n is compact or sequentially compact, if and only if it is closed and bounded. Well one direction we have already seen, sequentially compact sets must be closed and bounded; the other way is easy, the other way is easy, what you do is proof let A be, not A.

Let I 1, I 2 dot dot dot I n be intervals in R, closed and bounded intervals of course, closed and bounded intervals, such that k is subset of I 1, I 2 dot dot dot I n, the Cartesian product. Because K is bounded, I can find intervals closed and bounded intervals in R, such that K is the subset of the product of these intervals I 1, I 2 dot dot dot I n.

Now, by the Heine Borel theorem in the real numbers, each one of these I 1, I 2, I n are all compact; therefore the previous theorem will say that the product is compact in R n. Now, the moment the product is compact, K being a close subset must also be compact.

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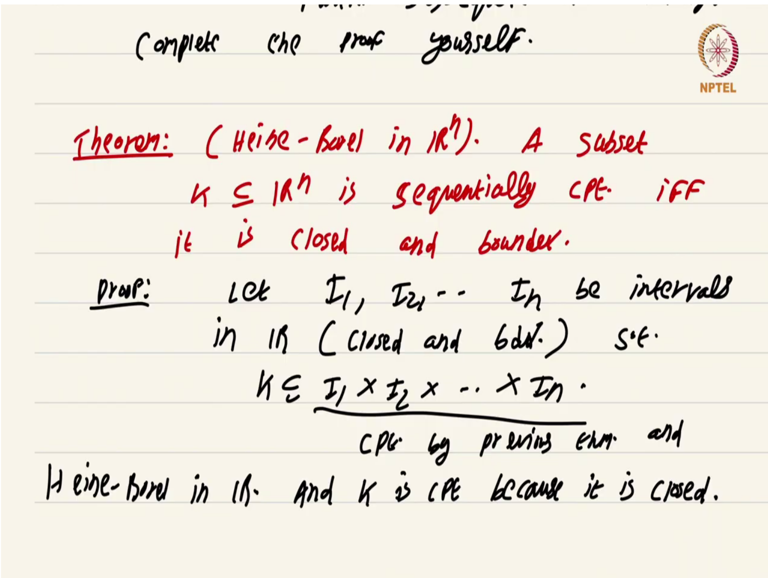

So, this is compact by previous theorem and Heine Borel and Heine Borel in R and K is compact because it is closed, because it is closed.

So, this completely characterizes the closed, sorry the sequentially compact subsets of R n; they are precisely the closed and bounded subsets of R n. So, the Heine Borel theorem is true in R n, though it is not true for a general metric space. So, the next endeavor is going to find out what the analog of Heine Borel theorem is; but that is the topic of an another video, let us just end this video with a very basic fact.

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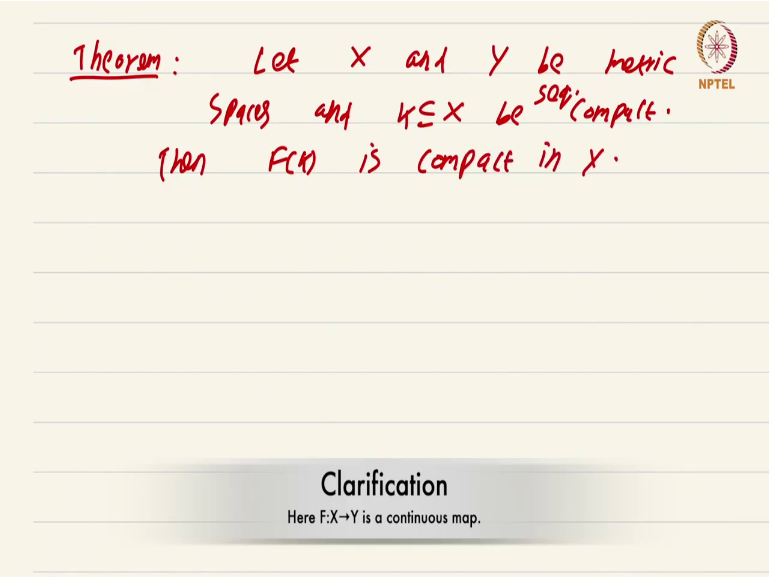

This is something that you should have anticipated.

Theorem; let X and Y be metric spaces, be metric spaces and K subset of X be compact sequentially compact of course; I keep forgetting to write this, then F of K is compact in Y.

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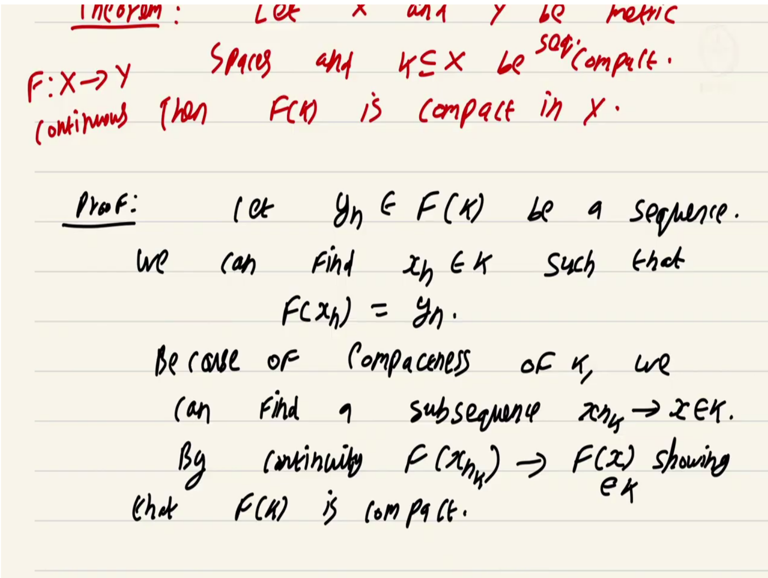

The proof is as easy as what it was on R, let y n in F of K be a sequence be a sequence. Then we can find we can find x n in K, such that F of x n converges to; not F of x n converges to sorry about that, I jumped a step, such that F of x n is equal to y n. Because these y n‘s are all coming from the image of the set K, we can find preimages x n; these are not unique, we can find some x n.

Then because of compactness, because of compactness of K, we can find we can find a subsequence x n k converging to x in K, ok. By continuity by continuity F of x n k converges to F of x showing and this F of x is of course in K, because x’s in K showing that F of K is compact.

So, the proof was utterly straightforward and exactly the same proof that we saw in the case of real numbers, ok. I just realized I glossed over an important thing, sorry about that; you must have been very confused for the past few minutes. I forgot to say let X and Y be metric spaces, K subset of X be sequentially compact and F from X to Y continuous.

Somehow I missed the crucial thing in this theorem; you have a continuous map from X to Y, then the image of a compact set is compact, sorry about that.

So, this concludes this introductory video on compactness, we will explore in depth in the coming videos. This is a course on Real Analysis and you have just watched the video on Compactness.