Real Analysis II
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Lecture - 5.1
Completion

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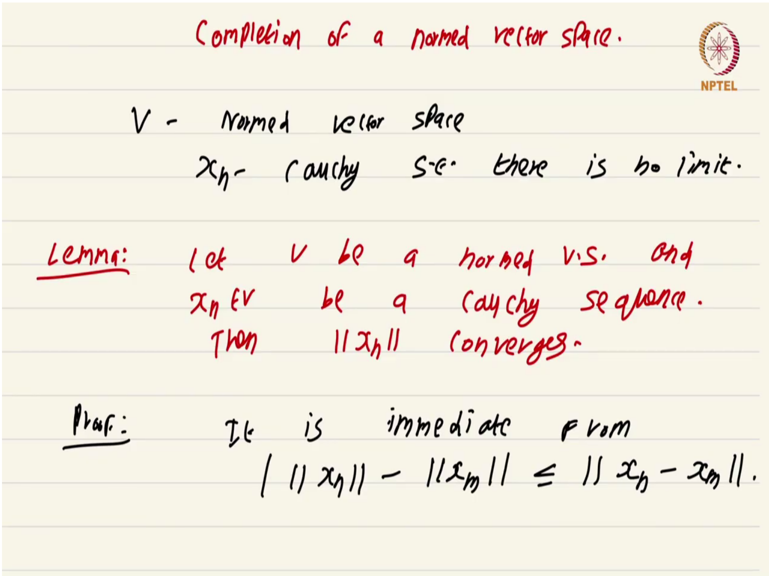

In this video I am going to sketch a proof of how we can start with the normed vector space and construct a complete normed vector space that in some sense contains the normed vector space we started off with. This essentially shows that any normed vector space can be completed. In particular, the construction I am about to sketch can be used to start with q and get r. The basic idea of how you go from q to r and how you complete a normed vector space are quite similar.

So, the situation we are in is, we have a normed vector space V, but it could possibly be incomplete. What that means is, there could be a Cauchy sequence x n Cauchy, such that there is no limit.

So the question is we have to somehow add points to the vector space V in order to make sure that every Cauchy sequence converges to a point in V. Now, what is that point you would add corresponding to this Cauchy sequence which is not converging? Well, what could be more natural when representing that particular point that we need to add by the Cauchy sequence itself.

So this might seem like an extremely bizarre way of doing it, but if you think about it for a moment, it is extremely natural. We need to complete it by adding points and these points correspond to Cauchy sequences in the original space V that do not converge. Therefore, we add that Cauchy sequence itself as the limit.

Now, one issue that might arise is that there are many Cauchy sequences that converge to the same point. If you had a vector space V, normed vector space V, there could be multiple Cauchy sequences, that could converge to the same point.

Furthermore, we need to sort of define a norm for this point that we are adding. So, let us do this step by step. First, let us take care of the norm by the simple lemma. Let V be a normed vector space, normed vector space and x n in V be a Cauchy sequence be a Cauchy sequence. Then norm x n converges ok.

So the proof of this is a rather trivial. This proof immediately follows. Proof it is immediate from absolute value of norm x n minus norm x m is less than or equal to norm of x n minus x m. From this it is immediate that this sequence norm x n is a Cauchy sequence and therefore, it must converge because r is complete ok.

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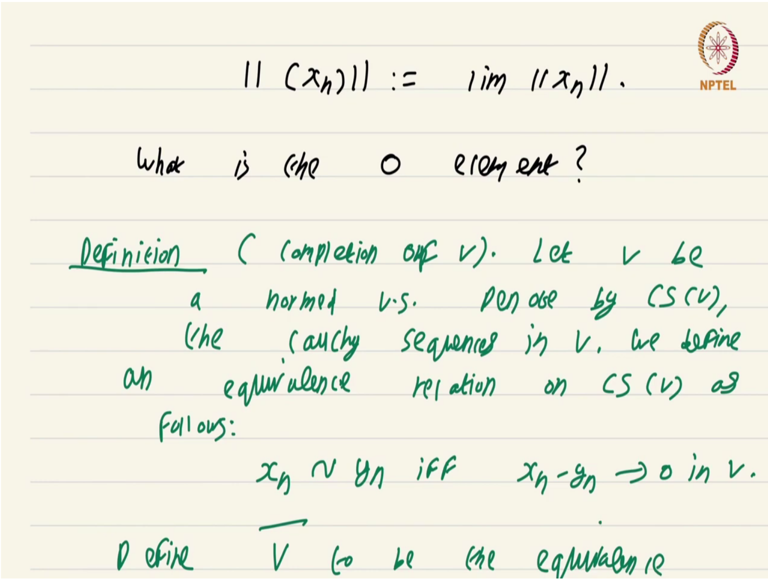

So, what this lemma suggests is that once you add these Cauchy sequences, let us suppose you denote I mean you going you are going to essentially define the norm of this Cauchy sequence to be nothing but limit of norm x n ok. That is what this lemma suggests.

Now, another question we have this new vector space V that has some bizarre elements, they are Cauchy sequences, what is the 0 element, what is the 0 element? Ok. Now, the 0 element should correspond to Cauchy sequences that converge to 0, but they are many of them and which one do you choose.

Well, we are going to choose in some sense all of them. So, let me now formally state the definition of the completion and I will sketch a proof that this definition is in fact makes sense. I mean, I am going to define various things and they may not be well defined. I will sketch a proof that all the things that I am claiming are well defined are in indeed well defined, completion of V ok.

So, the setup is let V be a normed vector space, normed vector space denote by CS of V the Cauchy sequences the Cauchy sequences in V. So, what we are doing is we are taking all the Cauchy sequences and putting it inside a basket and calling that basket CS of V.

Now as we have observed there are multiple Cauchy sequences that converge to the same point. So, we need to identify certain Cauchy sequences and the perfect way to do it is to use equivalence relations. We define an equivalence relation on CS V as follows. The sequence x n is related to the sequence y n if and only if x n minus y n converges to 0 in V ok.

The fact that this is a Cauchy sequences, I mean the fact that this is an equivalence relation is utterly trivial, so I am not even going to bother mentioning that anymore ok.

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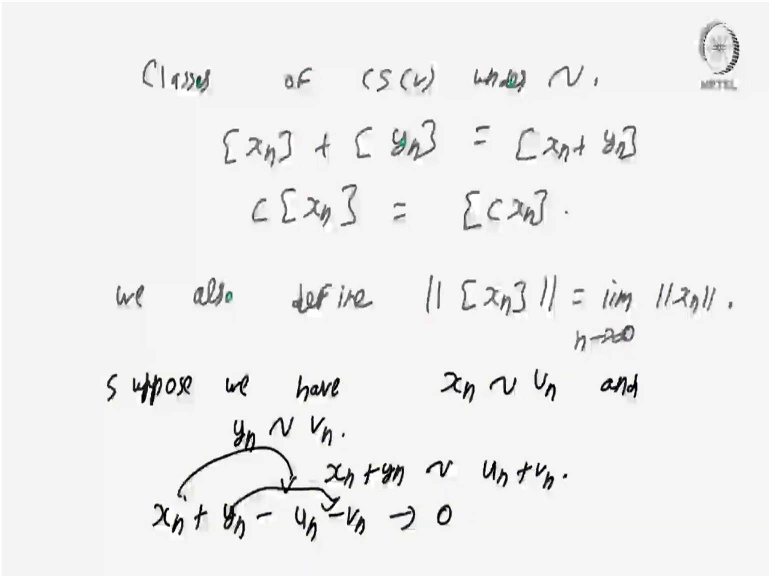

Now, the crucial definition. Define the completion of V which I am going to denote by V closure as a nice mnemonic. Define V closure to be the equivalence classes the equivalence classes of CS V under this equivalence relation ok.

Now, I am going to denote a equal an equivalence class like this ok. Now, we need to define, we need to define the operations under CS V that make it a vector space and those operations are rather straightforward. We define the equivalence class of x n plus the equivalence class of y n to be as you can guess x n plus y n the equivalence class of x n plus y n.

Similarly, scalar multiplication is dealt with in an exact same way. The equivalence class of C times, I mean the element corresponding to C times the equivalence class of x n is nothing but the equivalence class of C x n ok. We also define we also define we also define the norm of x n the equivalence class of x n to be limit norm x n, n going to infinity ok.

Now, there are number of trivial checks that need to be done to ensure that we have not done anything illegal in this definition ok. The fact that it is an equivalence class is trivial I am going to leave it to you. First, we have to show that these operations that we have defined x n plus y n and C x n, they are independent of the representative.

So, suppose we have, suppose we have x n is related to u n and y n is related to v n ok. Suppose we have we are taking two distinct representatives of the single equivalence class box x n. Now we have to show that the output that you get this x n plus y n actually no necessity to put a box. We have to show that this x n plus y n is related to u n plus v n. That will show that our operation is independent of the choice of representative.

But that is easy because x n plus y n minus u n minus v n converges to 0, because x n minus y and u n converges to 0 and y n minus v n converges to 0. Therefore, x n plus y n minus u n minus v n converges to 0. So, the addition operation at least is independent of the choice of representatives. In an exactly similar way, we can check that the scalar multiplication is also independent of the choice of representative.

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Showing $\bar{V}$ is a v.s. is just a long but easy check.

$$\|\{x_n\}\| = 0 \iff \|x_n\| \to 0 \iff x_n \to 0 \iff \{x_n\} = 0.$$

$$\|c\{x_n\}\| = |c|\,\|\{x_n\}\|.$$

$$\|\{x_n\} + \{y_n\}\| = \|\{x_n + y_n\}\|$$

$$\lim \|x_n + y_n\| \le \lim \|x_n\| + \lim \|y_n\|$$

$$= \|\{x_n\}\| + \|\{y_n\}\|.$$

Now, once you have these operations, showing this V closure is a vector space is just a long but easy check. You have to check each and every one of the stupid axioms of a vector space ok. So, we have at least constructed a vector space of Cauchy sequences or rather equivalence classes of Cauchy sequences.

We have defined a norm, we have to now check that, in fact this is a norm. So again, these are all straightforward easy checks, let me just do a few of them. Suppose, you have that the norm of a particular equivalence class is 0 ok. Now this will happen if and only if norm x n converges to 0, because that is how that is how the norm in this complicated space of equivalence classes was defined.

You just take norm x n and take limit n going to infinity. Now this will happen if and only if x n converges to 0 right, and this will happen if and only if this box x n is 0. So, this chain of equivalences proves that the norm of an element in our bizarre space V bar is 0 if and only if the representative, that is box x n itself is 0 ok.

Now, similarly you can show that I mean it is rather easy to show that norm of C box x n is equal to mod C norm of box x n and this is an easy check ok. Now coming to the triangle inequality what we have to do is, we have to consider norm of box x n plus box y n, but the operation means that this is nothing but norm of box x n plus y n ok. Which is nothing, but limit norm x n plus y n.

That is the definition of the norm in this vector space, which is less than or equal to limit norm x n plus limit norm y n, by the triangle inequality which is equal to norm of box x n plus norm of box y n.

So, this shows that the triangle inequality is satisfied by our definition of norm on the space V bar ok.

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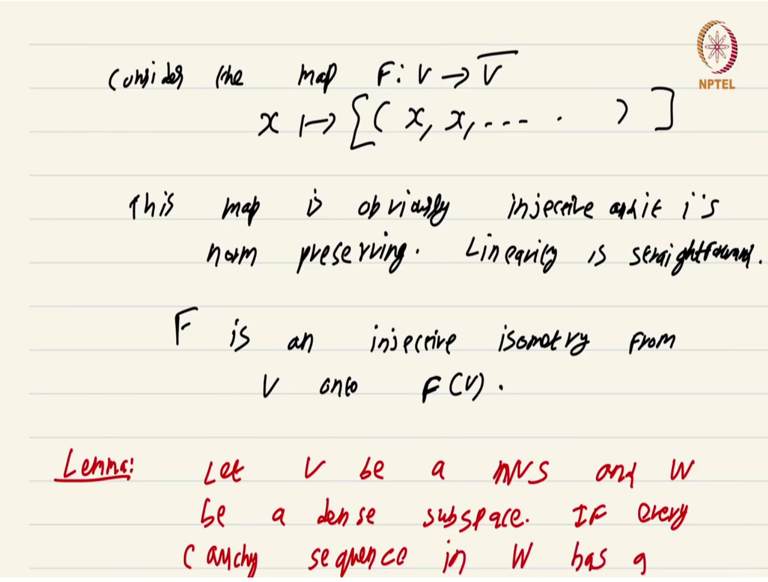

Now, consider the map consider the map, F from V to V bar ok, and the map is given by x maps to as you can guess the constant Cauchy sequence x x x x x. I should put a box around it for it for the definition to be 100 percent precise.

Now, this map is obviously injective. This map is obviously injective and it is equally obvious that it is and it is norm preserving that is the norm of x is the same as norm of box of x which is obvious ok. Linearity is straightforward. Linearity is also straightforward. I want you to do it, it is just a 30 second check ok.

So what this shows, is that this map F is an injective isometry from V onto F of V ok. So, what this shows is that V is sitting inside V closure as an isometrically embedded copy. So essentially, we have extended the space V to this larger space V bar or V closure or whatever you want to call it and V is sitting in there exactly as a copy that is in the guise of F of V in a different disguise. It is sitting in there as F of V.

Now, we still have to show that this bizarre space V bar is actually a complete normed vector space and that is going to be taken care of by the next lemma. The next lemma relates density and completeness ok. What it says is the following; let V be a normed vector space. I am just going to abbreviate it as NVS and W be a dense subspace.

So, it is a subspace of V whose closure is equal to V ok. If every Cauchy sequence every Cauchy sequence in W has a limit in V, then V is complete.

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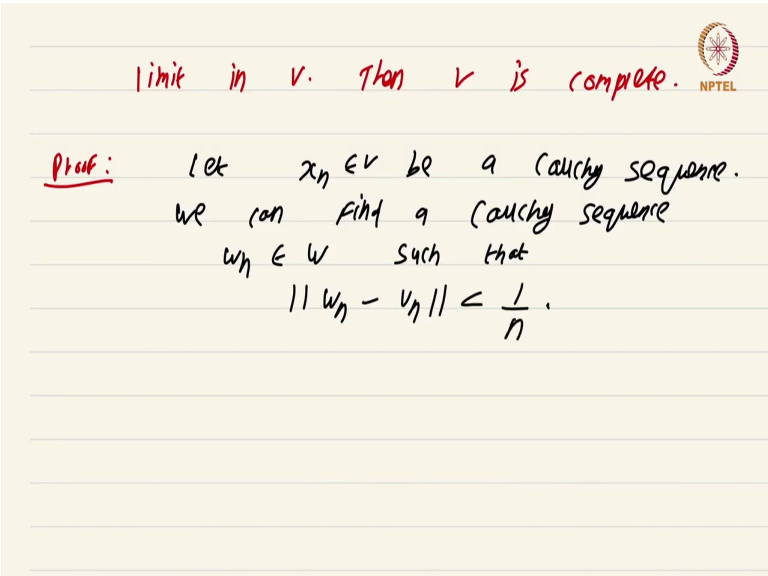

So, the setup is as follows; we have a normed vector space V, and W is a dense subspace. If every Cauchy sequence in W has a limit not necessarily in W, but in V then V is actually complete ok.

So, the proof is again not so difficult. Let x n in V be a Cauchy sequence. We have to show that this sequence has a limit in V. Now, because W is dense in V, we can find we can find a Cauchy sequence a Cauchy sequence W n in W, such that norm of W n minus V n is less than 1 by n ok. Now, if you want if you want to show that ok.

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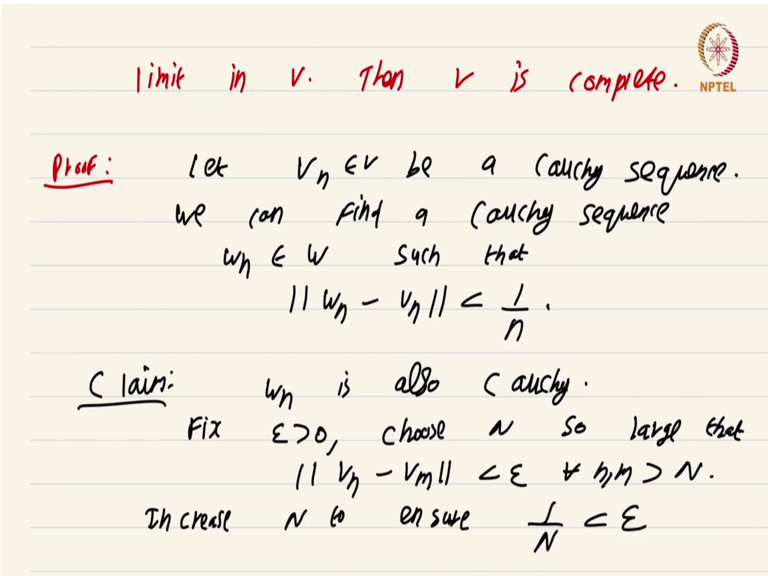

Just a second, I sort of made a slight mistake, let us just call this V n for convenience, because I have use V n later, in the beginning I said x n. So that makes no sense ok.

So, if you want to show that V n is convergent, we use the fact that whenever you have a Cauchy sequence in W it converges. So, what we are trying to do is we are trying to construct a Cauchy sequence W n in W, which would converge to the same point that V n would converge to, but W n must necessarily converge therefore, V n must also converge that is the basic logic ok.

Now, to do this we have to first claim W n is also Cauchy ok. Now, how do you see this well fix epsilon greater than 0 choose capital N, so large that norm V n minus V m is less than epsilon for all n comma m greater than capital N. This is of course possible because V n is a Cauchy sequence. Increase N to ensure 1 by N is also less than epsilon ok.

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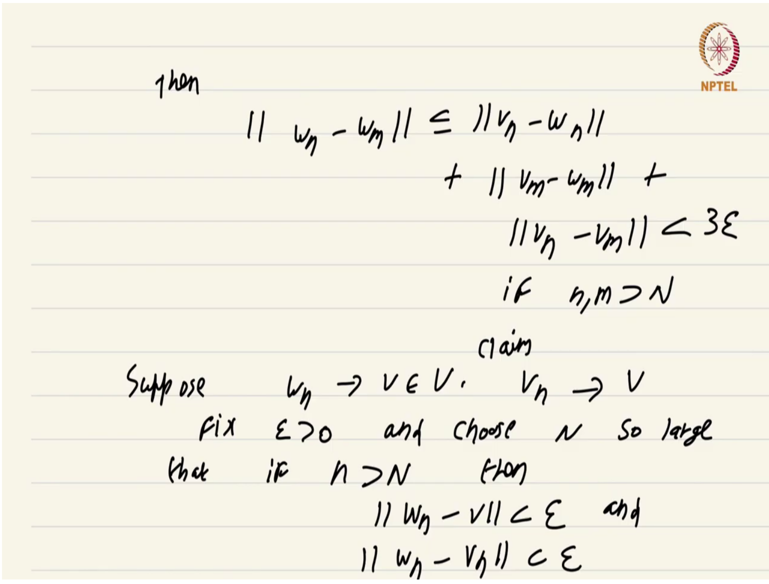

Then, observe that norm W n minus W m is by the triangle inequality less than V n minus W n plus V m minus W m plus V n minus V m. So, this is the three epsilon trick which we have done so many times and this is going to be less than or equal to 3 epsilon if n comma m is greater than N. In fact, this is going to be less than there is no need to write less than or equal to ok.

So, this proves that W n is a Cauchy sequence. So, suppose W n converges to V in V. By hypothesis there will be an element in V that converge that is the limit of W n. Now the claim is that V n also converges to V. This is the claim ok. So, again fix epsilon greater than 0 and choose N so large, that if small n is greater than N, then norm W n minus V is less than epsilon and norm W n minus V n is less than epsilon ok.

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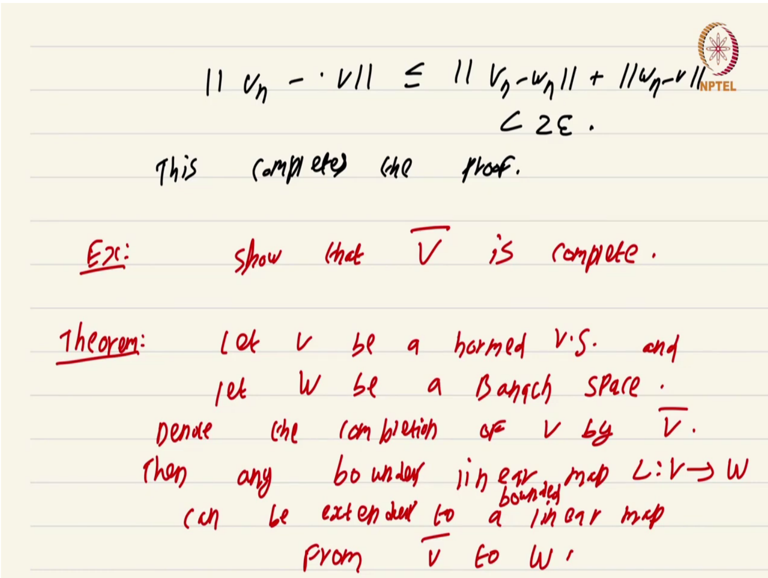

Then it is immediate, it is immediate that norm W n minus sorry norm V n minus V would be less than or equal to norm V n minus W n plus norm W n minus V will be less than 2 epsilon ok. So, this shows, this completes the proof ok.

So, what we have shown is the following; we have shown that if you start if you start with a vector space V and you have a dense subset W dense subspace W, such that W close such that every Cauchy sequence in W has a limit in V, then V itself is complete.

Now, final exercise for you show that this V bar is complete. And that will conclude the sketch of the proof that you can always complete a normed vector space. You can put it inside a larger sub larger normed vector space which is complete.

Not only that, once you solve this exercise, you will realize that this F of V is actually dense is actually dense in V. So, that essentially shows that you can embed any normed vector space as the dense as a dense subspace of a complete normed vector space.

Now, I am going to finish this video by one more theorem that is very useful in applications ok. So, what it says is the following; this is essentially about extension of linear maps to the completion. Let V be a normed vector space normed vector space and let W be a Banach space be a Banach space ok.

Denote the completion of V by V bar ok, then any bounded linear map L from V to W can be extended to a linear map from a bounded linear map of course, can be extended to a bounded linear map from V bar to W ok. If you start with the linear map from V to W it extends to the completion.

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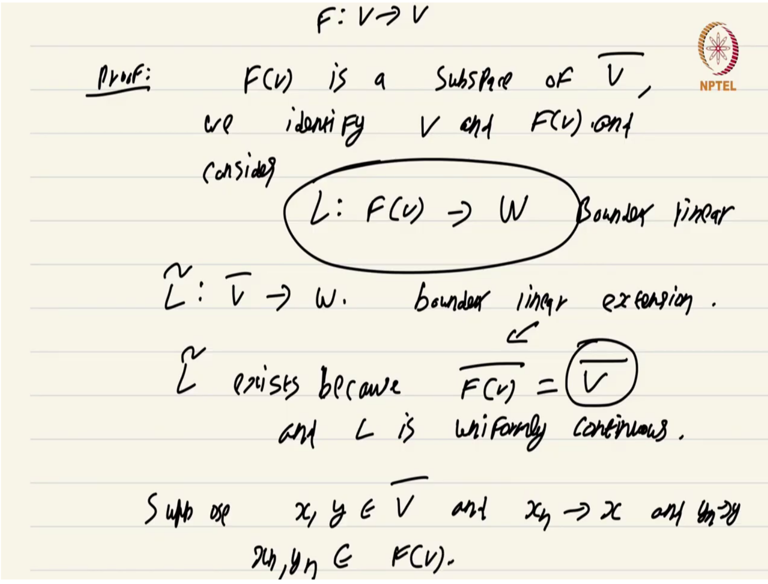

So, first of all before we even begin the proof, we have to understand what does, it mean that L extends. Well, since F of V is a subspace of W, we just identify we identify V and F of V, because anyway, so I must recall F is the map from V to V bar that we studied before.

So, since F of V is a subspace of not W is a subspace of V bar, we identify V and F V and consider L to be defined from F of V to W ok, bounded linear. So, there is a bit of unwinding to do and it is better that you do it by yourself, rather than me explaining what is happening there is nothing deep going on, but it might seem a bit confusing. Because we are starting off with the map L from V to W, now I am just saying that identify this isometric embedding of V inside V bar F of V with V and treat L as a map from F of V to W ok.

So, what the claim is that, if you start with this map L from F of V to W you can get a map L tilde from V bar to W that is the claim bounded linear extension it is a bounded linear extension ok. Now, how are we going to show this? Well this, L tilde exists, L tilde exists because of the previous exercise you already know that F of V closure is going to be V bar that is one, and L is uniformly continuous.

All bounded linear maps are uniformly continuous that we established previously when we studied the continuity of linear mappings between normed vector spaces. Because L is uniformly continuous and the fact that F of V closure is nothing but V bar, note here it is the closure ok and here it is just a notation. Since F of V closure is nothing but V bar, and L is uniformly continuous and uniformly continuous mappings extend continuously to the closure ok.

So, we have this map we have this map, L bar from a V bar to W. Note, this is the place where we use the fact that W is a Banach space. So, we get a continuous linear extension L tilde from V bar to W. The only thing that remains to be shown is that this L tilde is actually linear and that is rather straightforward.

So, suppose x comma y are elements of V bar ok, and x n converges to x and y n converges to y, where x n and y n are coming from they are coming from V ok, which I am identifying with F of V tacitly ok. So, take two points x comma y in the completion V bar and consider sequences x n converging to x and y n converging to y, where x n and y n are coming from V which let me just for clarity sake just identify it already with F of V ok.

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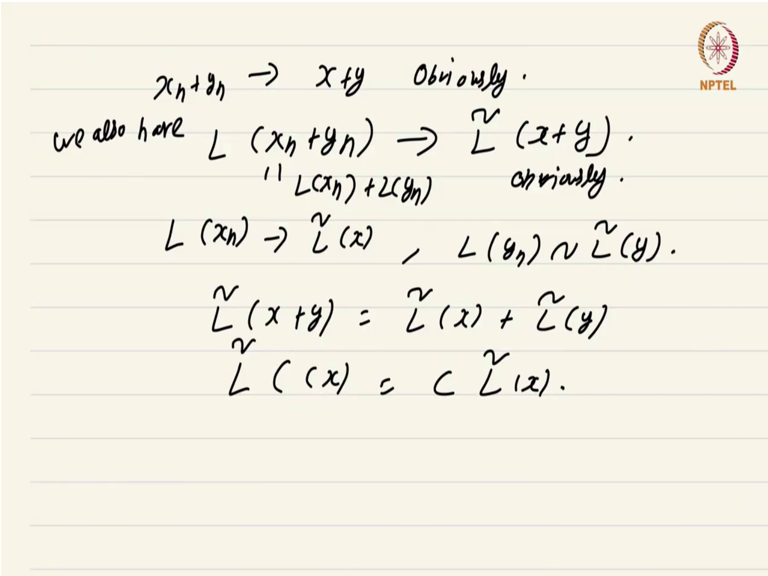

Then, x n plus y n converges to x plus y, this is obvious; obviously ok. Now, what we have to do is to show that L of x n plus y n L of x n plus y n converges to L tilde of x plus y ok. This is also obvious sub we also have we also have L of x n plus y n converges to L tilde of x plus y; obviously, this also this follows from the very fact that L tilde is an extension and it is a continuous extension and since x n plus y n converges to x plus y, L of x n plus y n must converge to L tilde of x plus y ok.

Now, similarly we have L of x n converges to L tilde of x and L of y n converges to L tilde of y. This is again by the fact that L tilde is a continuous linear extension. Putting all this together we get that L tilde of x plus y is equal to L tilde of x plus L tilde of y this is because L is linear. So, this part L of x n plus y n is just L of x n plus L of y n ok.

So, this concludes the fact that L tilde of x plus y is L tilde of x plus L tilde of y. So, we have essentially exploited the linearity of the map L on the space F of V and somehow translated to L tilde by passing to limits. Similarly, you can show that L tilde of C x is C L tilde of x exactly in the same way.

This will prove that this extension L tilde is linear. It is already continuous from the fact that we got L tilde by using a continuous extension theorem that we saw in a previous video. So, this was a somewhat longer module, but work on it. It is not that difficult it consists of a number of trivial checks that I have left for you intentionally, so that you will have a deeper understanding of what is happening.

So, the moral of the story is, you have for a given normed vector space. You all always have a completion and all linear maps to Banach spaces from the normed vector space extend to be bounded linear maps in the completion also.

This is a course on Real Analysis and you have just watched the video on the Completion of a Normed Vector Space.