Real Analysis II
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Lecture - 4.3
Completeness of B(X, Y)

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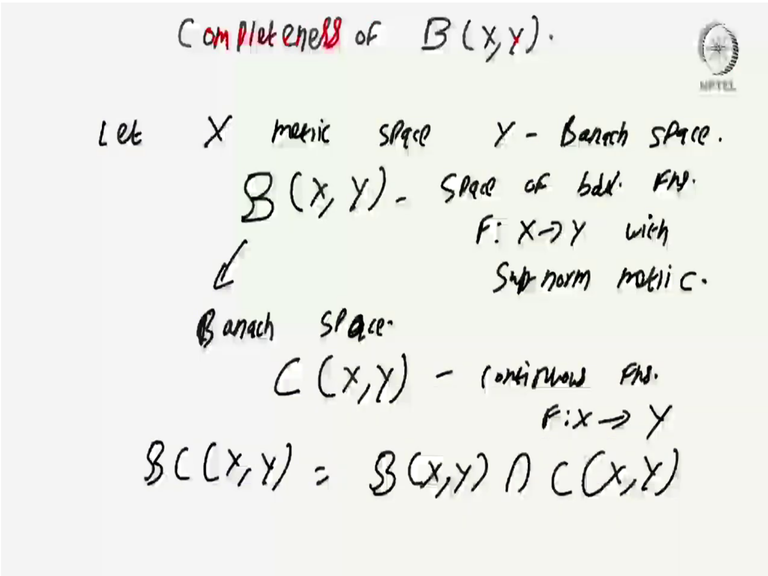

In this video, we are going to deal with a very important theorem the fact that the space of bounded functions from a metric space into a Banach space is actually complete. So, let me just set the stage before stating the theorem, in this video X would be a metric space, Y is going to be a Banach space you can just take Y to be R that is the most common scenario.

So, X is a metric space, Y is a Banach space. And recall that B X, Y this fancy B if it looks like some sort of snake just do not worry about it is just I am writing it very badly. So, let me just write it somewhat legibly some B X, Y, this is nothing but space of bounded functions F from X to Y with sup norm with the sup norm metric ok with the sup norm metric, this is a complete. This is a Banach space; this is a Banach space that is the claim ok.

So, let me just formally state it as a theorem, but before that let me add one more thing not only is this is a Banach space, you can also consider the space C of X, Y space of continuous functions F from X to Y again with the same sup norm metric I am not going to write that down. This C X, Y, this C X, Y is not necessarily a subset of B X, Y because you have no such results saying that a continuous function is going to be bounded.

There is no such result unless X happens to be compact or something. So, we consider this BC X, Y – this is just B X, Y intersect C X, Y ok. So, this is the space of functions that are both bounded as well as continuous. This BC X, Y is also complete that is the entire result ok.

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Theorem. Let X be a metric space and Y be a Banach space. Then $B(X,Y)$ and $BC(X,Y)$ are Banach spaces.

Proof: Let $F_n \in B(X,Y)$ be a Cauchy sequence.

$\forall x \in X$, $F_n(x)$ is Cauchy.

Call $\lim_{n\to\infty} F_n(x) =: F(x)$.

So, let me just state the result now. Theorem, let X be a metric space, and Y be a Banach space be a Banach space, then B X, Y, this should actually be that fancy B, B X, Y and BC X, Y are Banach spaces are Banach spaces ok. The fact that these are non-vector spaces that we have already seen and you must have checked all the boring axioms of vector space for both B X, Y and BC X, Y. If not do so now it is rather easy to do. So, the key fact is to show completeness ok.

So, let us first show the completeness of B X, Y. So, let F n in B X, Y, B X, Y be a Cauchy sequence be a Cauchy sequence ok. Now, an exercise for you which is going to take you not more than a minute or 2, for each x in X F n x is Cauchy. This is, almost by the definition of the sup norm metric. So, I am going to leave that to you to check each sequence F n x is actually Cauchy ok. Call limit n going to infinity F n x to be equal to F of x ok. So, rather this is just by definition F of x is defined to be the limit as n goes to infinity of F n x ok.

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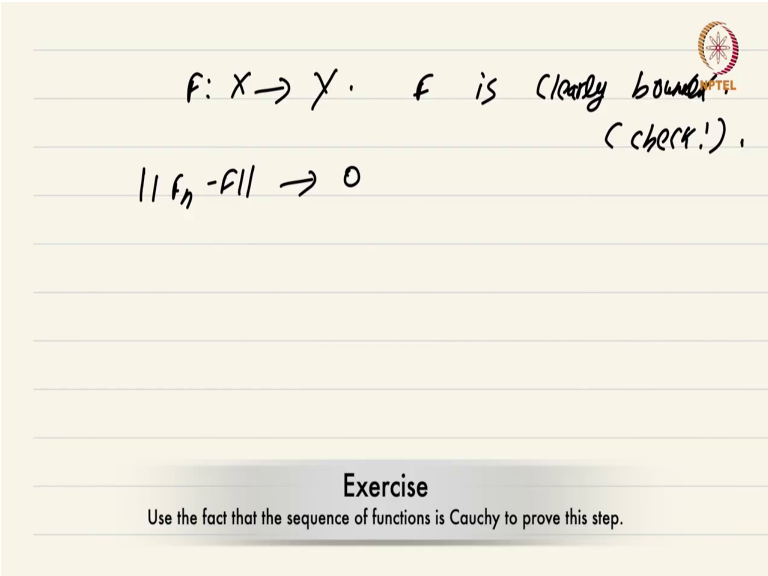

This way we get a function F from X to Y, we get a function F from X to Y. First is F is clearly bounded, F is clearly bounded. Again this is all these are all easy checks. So, I am going to be very fast, and leave it to you. So, again just check this that F is bounded ok.

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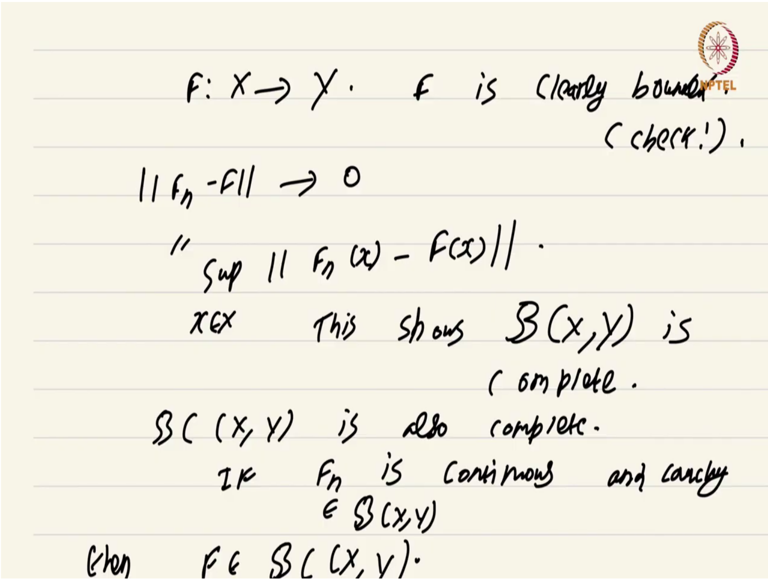

And norm F n minus F clearly converges to 0 simply because of the way we have defined F, F is just the limit of F ns F limit of F n x. And this is nothing but supremum X in X supremum X in X norm of F n x minus F x ok. So, this must converge to 0 ok. So, this shows that B X, Y is complete ok.

So, what have we achieved by all this? We have now shown we have now shown that B X, Y is a Banach space. Now, what we have to further show is that BC X, Y is also complete is also complete ok. Now, we have already seen that any closed subset of a complete metric space is going to be a complete metric space. So, all we have to show that is if F n, is continuous so rather if F n in B X, Y is continuous and Cauchy, then F is also an element of BC X, Y.

So, what we are essentially showing is that the limit under the sup norm of continuous functions from X to Y is also going to be continuous. And the proof of this is something that we have already seen. It is a very similar to showing the fact that the limit of a uniformly convergent sequence of functions is continuous.

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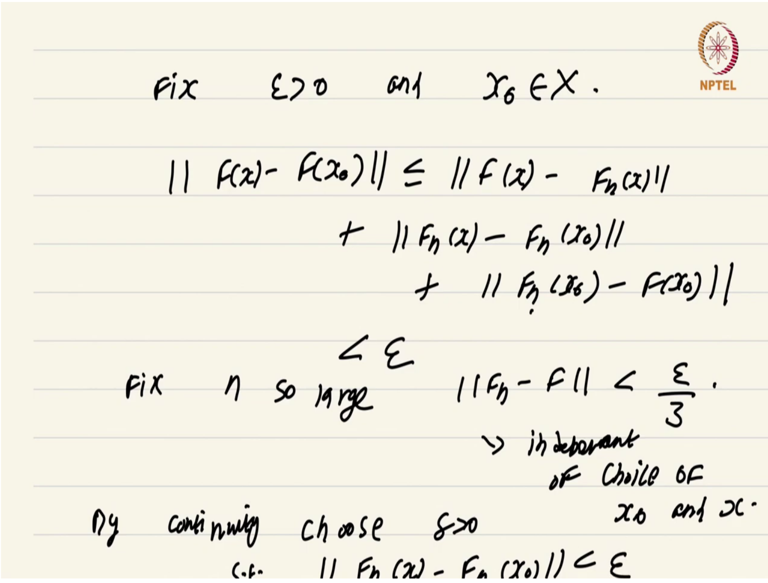

So, what you do is fix epsilon greater than 0, and choose x naught an element in X ok. Then observe that norm of F of x minus F of x naught is less than or equal to norm F of x minus F n x plus norm of F n x minus F n x naught plus norm of fn x naught minus F of x naught ok. So, what this essentially shows is each one of these terms F of x minus F n x, F n X minus F n x naught, and F n x naught minus F x naught, all of these can be made individually less than epsilon ok.

And this is exactly the same argument that we use for showing that uniformly convergent sequence of continuous functions must be continuous. So, let us just for our completeness sake, let us just write down just a little bit of the details. What you do is fix n so large fix n so large that norm F n minus F is less than epsilon by 3. This can be done because F ns converge to F in the sup norm ok.

Now, this is independent, this is independent, this is independent of choice of x naught and x, this is completely independent of the choice of x naught x. In fact, earlier when we did this, this part supremum x, supremum x in X, norm F n x minus F x, this goes to 0. Even here you are actually under the rug you are actually using the fact that you have some sort of uniform convergence because we are using the sup norm.

So, when you check the details, when you check that this function F is bounded and that norm F n minus F converges to 0, please check that you are actually using some sort of uniform convergence that is the reason why we can actually make this independent of the choice of point.

You cannot I mean at the outset I just said that F of x is just the limit of F n x that does not guarantee that this will be less than epsilon. It could happen that different points you have to go further and further in the sequence to make norm F n x minus F x less than epsilon, but that will not happen because we are putting the sup norm metric.

So, please carefully go through the details when you check this ok. So, coming back to the situation we are in. We can make norm F n minus F less than epsilon by 3 independent of the choice of x naught and x that immediately will show that this first element and this last element are both less than epsilon by 3.

So, the only thing we have to take care is the middle element, and that middle element can be taken care of by continuity. So, by continuity choose delta greater than 0 such that norm F n x minus F n x naught is less than delta sorry is less than epsilon is less than epsilon.

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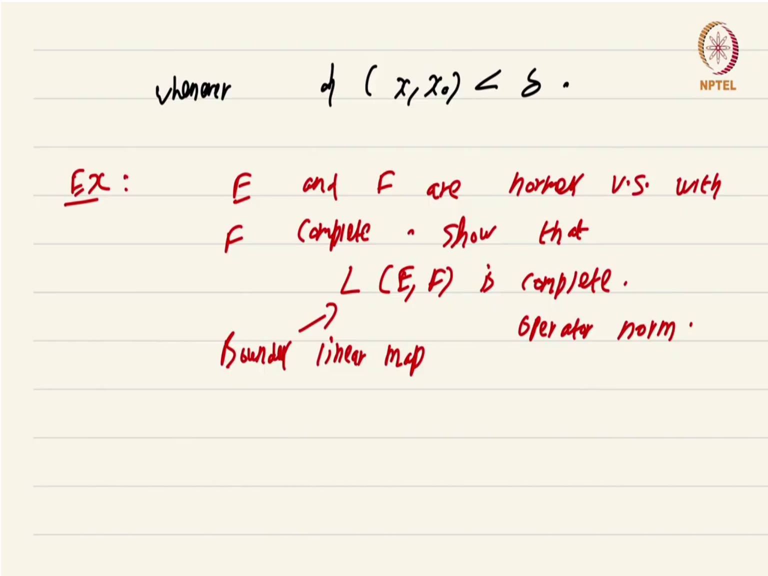

Whenever d x, y sorry whenever d x x naught x x naught is less than delta ok. So, what this is saying is the middle term norm F n x minus F n x naught you can control by using the continuity of F n ok. So, note here we are fixing n very large that is why this works. Fix n very large.

From that n, you get the F n, the first and the last terms you control by the fact that you have some sort of uniform convergence, and the middle term you control by the continuity of F n. So, this way we get a delta that works in the definition of continuity for F.

As this is an argument that is quite easy and familiar to you, I am being fast and I am skipping a lot of steps, please fill in all the details this is especially important because this is the central one of the central results on metric spaces that the space BC B of X, Y and BC of X, Y are both complete metric spaces. I am going to leave you with one important exercise that will really help you understand what is happening. It is a similar to what we have done.

So, what you do is e and F are normed linear spaces normed vector spaces with F a Banach space if F is complete ok. Show that; L E, F is complete. The space of linear mappings from E to F is complete with operator norm operator norm. Of course, I miss L E, F is space of bounded. Remember that it is bounded linear maps. From E to F is complete with the operator norm. Solve this exercise, and you will have a good understanding of what is going on.

One more remark, in the notes, I have given a long exercise taken from the very famous textbook by Simmons I think it is called Introduction to Topology and Modern Analysis or something like that. It is a classic textbook. This is an exercise that shows you that given a metric space you can construct a completion.

In fact, you can put it inside a complete normed vector space that is a Banach space. Please solve that exercise. In the next video I am going to show you how to complete a normed vector space, there the proof is lot simpler. So, please do solve that exercise. It is a step by step exercise. You will have to fill in a lot of details.

This is a course on Real Analysis, and you have just watched the video on Completeness of B X, Y.