Real Analysis II
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Lecture - 4.2
Completeness Continued

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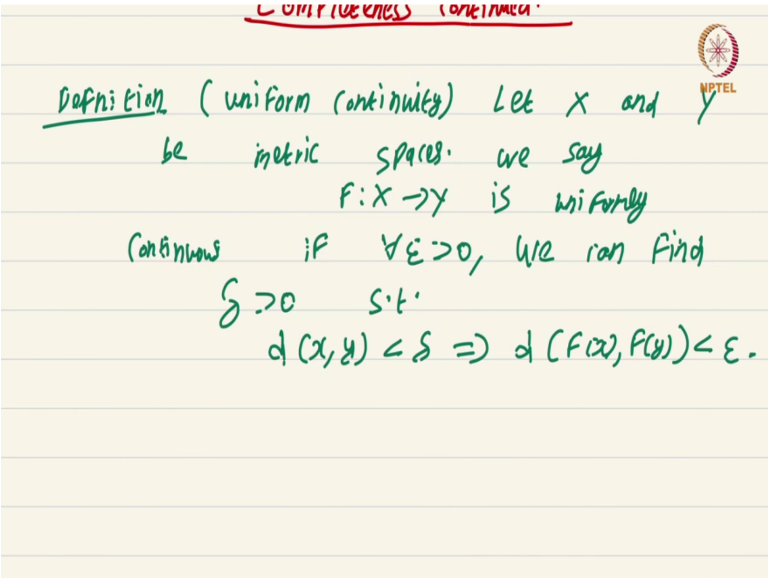

In this video we are going to continue our exploration of the Notion of Completeness, we begin with the definition that should be very familiar to you from our study of topology on the real line. This is the definition of uniform continuity and the definition is exactly what you would expect if someone were to tell you please define uniform continuity for me this is the definition that you should come up with yourself even if you have never seen it before.

So, the definition is as follows let X and Y be metric spaces; be metric spaces. We say F from X to Y is uniformly continuous; is uniformly continuous if for each epsilon greater than 0, we can find; we can find; we can find delta greater than 0 such that d x, y less than delta implies d F of x F of y is less than epsilon. If this definition looks vaguely familiar to you, well that is because it is exactly what you would expect.

What uniform continuity says is the following thing that the continuity for checking continuity given any choice of epsilon and given a point x you must be able to find a delta that works in the epsilon delta definition, what uniform continuity requires us is this choice of delta does not depend on the point x in question. So, you must have whenever x and y are suitably close then you must immediately have F of x and F of y to be suitably close as well. So, the choice of epsilon the choice of delta is a function of only epsilon and is independent of the choice of point.

And as you can expect the notion of uniform continuity will behave well with respect to compactness that is any continuous function on a compact set will become uniformly continuous the exact same proof will work once we develop the machinery for compactness I will at that appropriate point leave it as an exercise for you to show that a continuous function on a compact set is automatically uniformly continuous.

So, I am not going to bother writing down more examples of uniformly continuous functions you have already seen several of them when we studied topology on R, rather what I am going to do is I am going to prove one important theorem that we already left as an exercise way back when we studied topology on the real numbers, but this time I am going to give a full proof because this is going to be really important for applications. So, that theorem is as follows.

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Theorem: let X be a metric space and let Y be a complete metric space. Let $S \subseteq X$ and $F : S \to Y$ be a uniformly continuous fn. Then we can extend F to a continuous map from $\bar{S}$ to Y.

Proof: let $x_n \in S$ a Cauchy sequence. $F(x_n)$ is also Cauchy. Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$d(x,y) < \delta \Rightarrow d(F(x), F(y)) < \varepsilon$$

Theorem, let X be a metric space X be a metric space and let Y be a complete metric space let Y be a complete metric space, let S subset of X and F from S to Y be a uniformly continuous function. So, what I am doing is I am considering a subset S of the metric space X I am giving it the subspace to metric I am treating S as a subspace of X and I have a uniformly continuous map from S to Y note Y is complete, no such assumption is made on X ok.

Now, here is what the result says. Then we can extend F to a continuous map to a continuous map from S closure to Y ok. In short any continuous map rather any uniformly continuous map from a metric space to a complete metric space extends in fact, to the closure ok.

So, let us prove this, the proof is exactly the same idea what we have seen before, but let us write down all the details ok. Now the essential idea that we are going to use is the fact that uniformly continuous mappings behave very well with respect to Cauchy sequences. So, what we are going to do is we are going to start with the Cauchy sequence x n and then we are going to see what happens to F of x n.

So, the claim is F of x n is also Cauchy ok. Well, why is this the case? Well, given epsilon greater than 0 there exists delta greater than 0 such that d x, y is less than delta implies d F of x F of y is less than epsilon. This is nothing but the definition of uniform continuity and since F is given to be uniformly continuous this is true of course, this is true for all x y for all x y in S ok.

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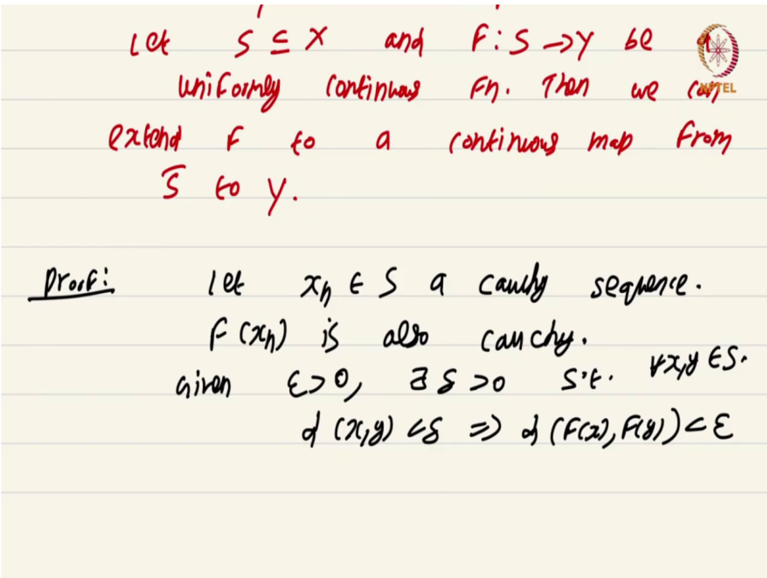

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Because x_n is Cauchy, $\exists N \in \mathbb{N}$

s.t. if $n, m > N$ then

$d(x_n, x_m) < \delta$.

This means $d(F(x_n), F(x_m)) < \varepsilon$

by u.c.

$\in S$

Suppose $x \in \bar{S}$ and $x_n, y_n \to x$

then observe z_n

$x_1, y_1, x_2, y_2, x_3, y_3, \ldots$

So, how does this help us? Well, because x n is a Cauchy sequence; because x n is a Cauchy sequence there exist capital N in the natural numbers such that if n comma m are greater than capital N then d of x n comma x m is less than delta ok. This is just the definition of a sequence being Cauchy applied to the case I am choosing delta to be the quantity the sequence must get close to eventually.

So, you can find a natural number N such that if n and m are greater than capital N then d of x n, x m is less than delta this is this should be trivial at this point. This means d F x n, F x m is less than epsilon this is by uniform continuity I will just abbreviate it as u. c. So, by uniform continuity what we have shown is that given any epsilon we can find a capital N so large that if n comma m are greater than capital N then d of F x n, F x m is less than epsilon ok.

So, this shows that we have Cauchyness being preserved by uniformly continuous functions ok. Now, suppose x is an element of S closure and x n and y n both converge to x coming from S. So, what I am saying is consider this set S and suppose we are in this situation where we have two sequences from S from S that converge to the same point x in S closure.

Then, observe that if I consider this new sequence z n I am not going to write down a formula for z n I am going to leave that to you, but z n is just the sequence x 1, y 1, x 2, y 2, x 3, y 3 and you get the big picture it is rather trivial how to write down a precise formula for z n I am going to leave it to you, this sequence z n is also Cauchy ok.

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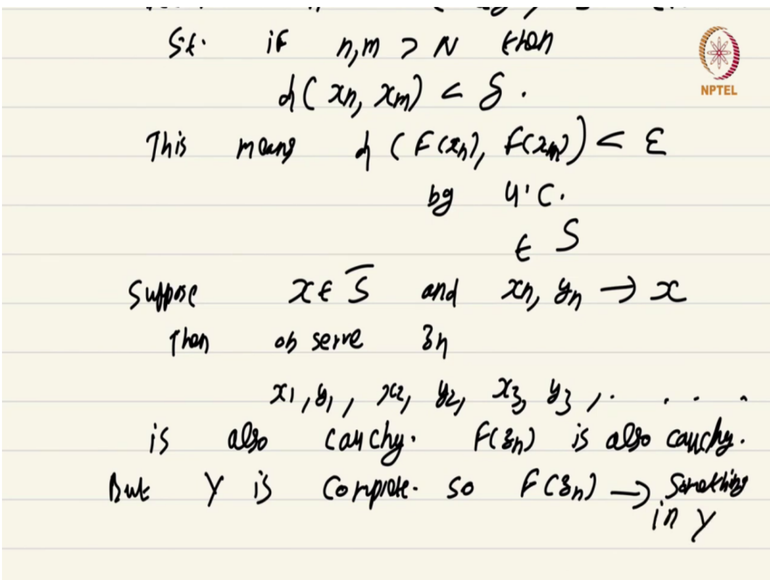

This is a rather easy check because both x n and y n converge to x you can prove in a line or two that z n must be Cauchy. Consequently F of z n is also Cauchy that is what we just established a few minutes ago, but F of z n is Cauchy and Y is complete, but Y is complete Y is complete. So, F of z n converges to some element in Y to something I do not care what it is something in Y ok.

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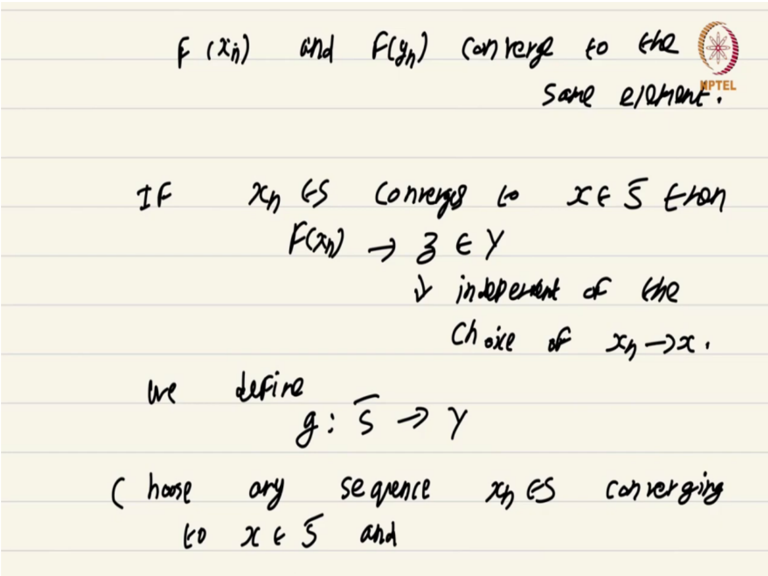

Well, why am I saying all this? Well, that just means F of x n and F of y n converge to that same thing converge to the same element simply because they are these both are essentially subsequence of F of z n. So, I am being a bit vague and imprecise here, but I trust that you are at a stage of mathematical maturity where you can make such vague statements quite precise in a matter of minutes ok.

So, what have we achieved? What we have achieved is if x n in S converges to S converges to x sorry in S closure then F of x n converges to an element z in Y and the z is independent of the choice of x n converging to x. It depends only on the point x in S closure it does not depend on the particular sequence you choose to approach this point x coming from the set S ok.

So, we can define we define g from S closure to Y as follows choose any sequence any sequence x n in S converging to x converging to x in S closure.

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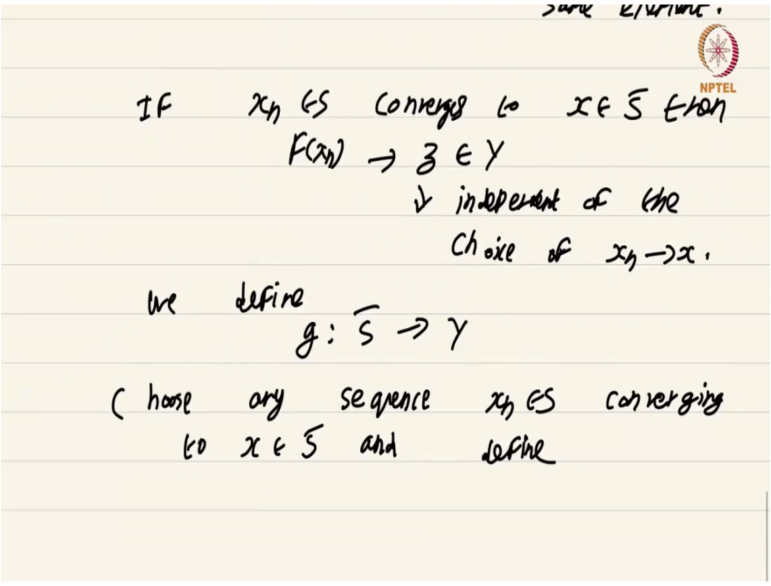

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$$g(x) = \lim_{n \to \infty} F(x_n).$$

If $x \in S$, we can consider the constant sequence x to see that $g(x) = F(x)$.

Fix $\varepsilon > 0$ and choose $\delta > 0$ coming from u.c. of F.

Let $x, y \in \overline{S}$ s.t. $d(x,y) < \frac{\delta}{3}$.

And define let me go to the next page just give me a moment define g of x to be limit n going to infinity F of x n. So, up until now the entire discussion has been to show that this g is well defined it does not depend on the choice of sequence x n we choose.

Furthermore if x is in S if x is in S, we can consider we can consider the constant sequence; the constant sequence x rather xxxxx repeat. So, I am just calling it the constant sequence x to show that or rather to see that g of x equal to F of x ok. What have we achieved? Well, what we have achieved is the following.

Given this function F from S to Y we have found out an extension g from S closure to Y. Now what remains to be seen is why is this function uniformly continuous sorry why is this function continuous. In fact, we are going to show that it is uniformly continuous, but ah the theorem just claimed that g is continuous.

So, we will again check the epsilon delta definition. So, fix epsilon greater than 0 and choose delta greater than 0 coming from uniform continuity of F ok. So, the rest of the argument is in fact, going to establish that g is uniformly continuous not just continuous. So, the choice of delta comes from the uniform continuity of the function f.

So, let x comma y be elements of S closure ok and such that d x, y is less than delta by 3. I am going to show that d F of x F of y is less than epsilon this is a standard argument that is that should be familiar to you by now, anyway let us run through the argument quickly once more.

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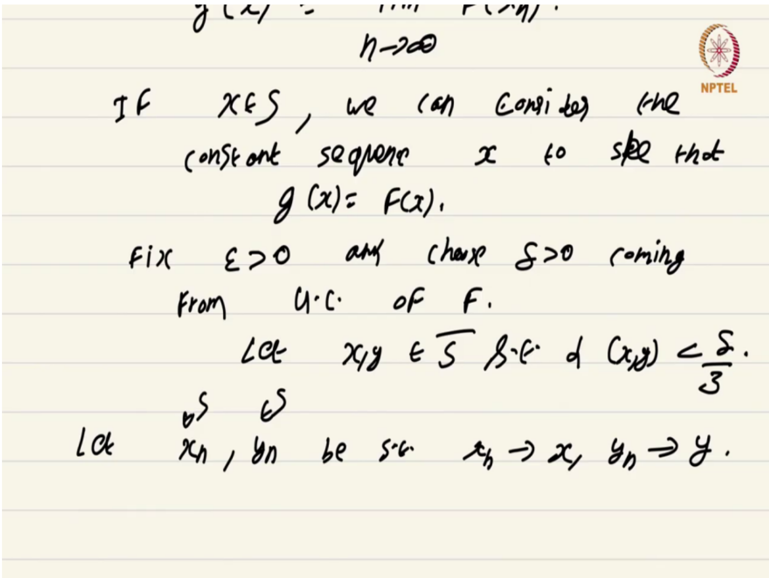

Let x n in S comma y n in S be such that x n converges to x, y n converges to y ok, you will understand in a moment why we chose this particular delta by 3 and not delta or delta by 2 why delta by 3 is crucial ok.

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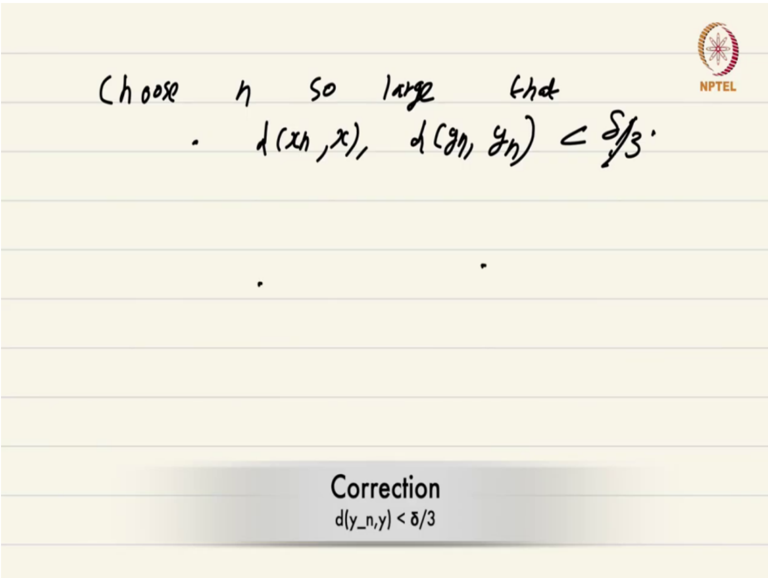

So, now, I am going to start a new page. Choose n so large that first of all d of x n comma x, d of y n comma y are both less than delta by 3 ok. Remember d of x y was less than delta by 3. So, if x is here, y is here.

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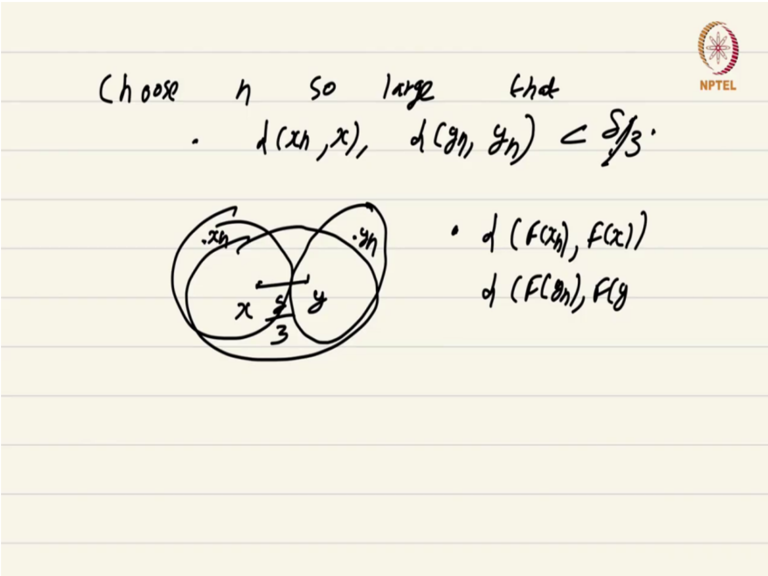

So, let me next not draw a primitive ball what I do is. I choose x and y close to each other ok the distance between x and y is at the max delta by 3. Now what I am saying is choose delta by 3 I mean these are very badly drawn figures, but it will give you an idea of what is going on. So, this if this is x this is y I am saying choose x n coming from here y n coming from here ok, that is part 1. Second step is d F x n comma F x, d F y n comma F of y.

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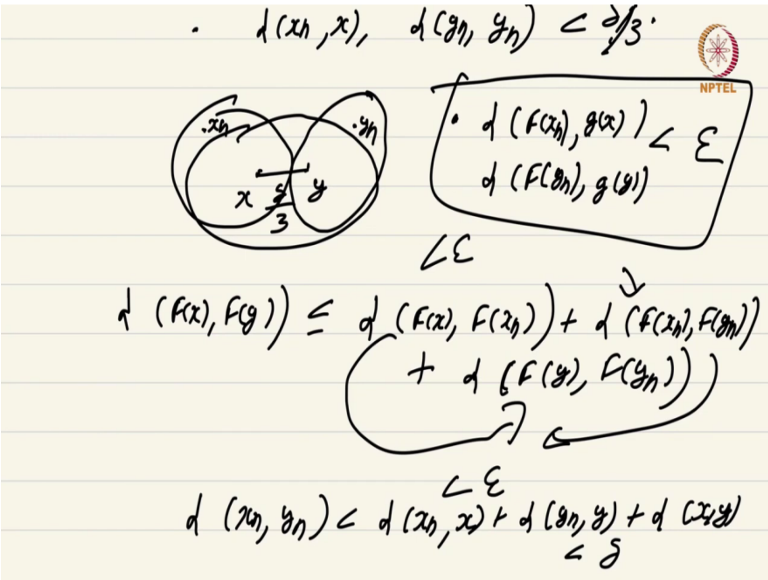

Sorry, there is a slight mistake here F of y does not make sense this is g this is g of x and this is g of y, because x and y are coming from S closure.

Let me just check whether I have not made a mistake here yes x and y are coming from S closure. So, you cannot really say F of x and F of y that does not make sense and anyway that is irrelevant to our purpose we are our interest is solely in showing that the distance between g of x and g of y is less than epsilon.

So, what you do is, you choose n so large that first of all d of x n comma x d of y n comma y are both less than delta by 3 and d of F x n, g x and d of F y n, g i g y both of these should also be less than epsilon both quantity should be less than epsilon.

Now we have ensured; we have ensured by our choices that d of F x F y is actually going to be less than 3 epsilon, why is that? Well, d of F of x F of y is going to be less than or equal to d of F of x, F of x n plus d of F of x n, F of y n plus d of F of y, F of y n ok.

Now, these two quantities d of F of x, F of x n and d of F of y, F of y n are both less than epsilon just by this. However, d of F of x n, F of y n is also less than epsilon simply because d of x n, y n is less than d of x n, x plus d of y n, y plus d of x, y which is less than delta. So, now, you should understand why we chose that particular choice delta by 3 at this point at this point why we chose delta by 3 and not anything else.

So, that is it this completes the proof d of F of x F of y is less than 3 epsilon ok therefore, we have that ok I made a slight error here.

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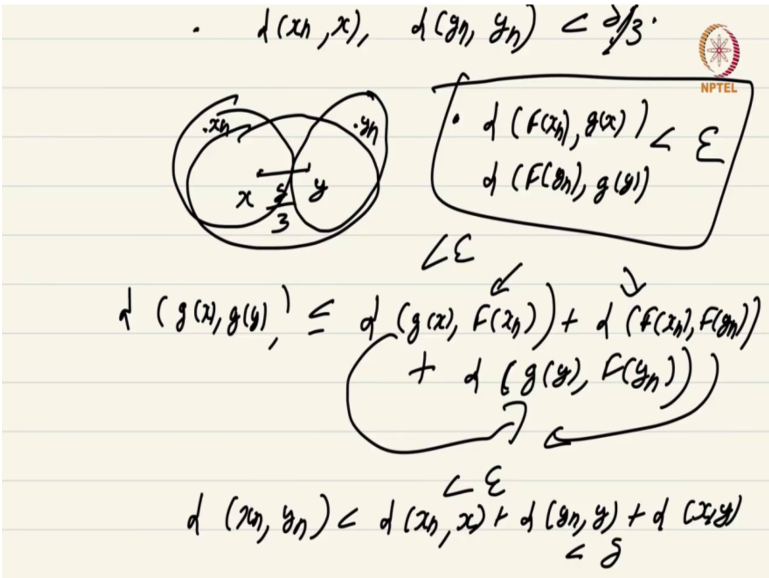

This sorry about this; this is g of x, g of y and again I must change certain fs to gs to be correct. So, usually what happens is the extension is also denoted by the same letter f, but here I have used I have used g. So, I must use g ok. Note that these at these places also I could have replaced F by g it really does not matter, because on the set S F and g agree ok. So, this concludes the proof that g is continuous in fact, uniformly continuous.

So, let me end this video by some remarks about the property of completeness with respect to topology ok.

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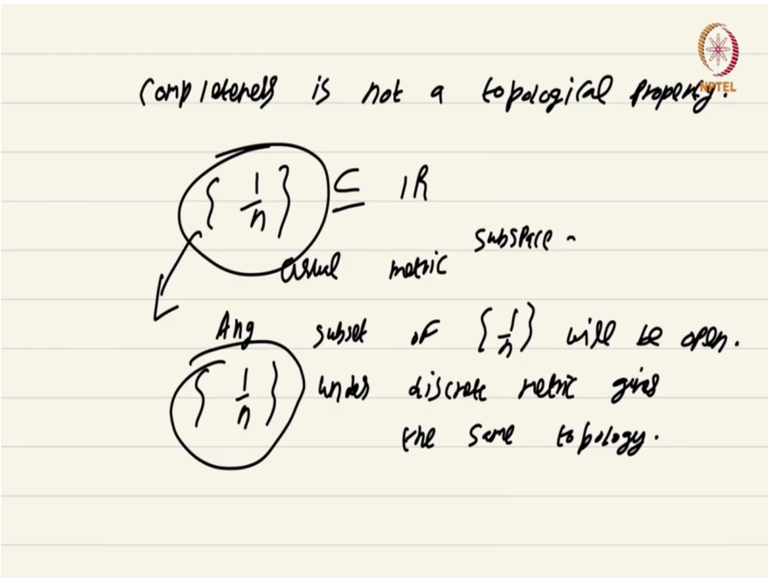

Completeness is not a topological property ok. Just like I mean what do I mean just like boundedness, boundedness is also not a topological property. I am not going to rigorously define what a topological property is that is going to be left for the future when you do a course on topology sometime in the sometime.

A topological property loosely speaking is a property on a metric space that can be characterized completely using open sets, this is just vague ok I am not being 100 percent precise. We have already seen that boundedness cannot be characterized completely just using open sets. In fact, you can find a metric space with two different metrics or rather a set with two different metrics that are equivalent in the sense that the open sets are same, but in one of them x is bounded in the other x is not.

We have already seen an example of this in an earlier video when we discuss equivalent metric spaces. So, being bounded is not a topological property right it is not a topological property I was right being bounded is not a topological property. What about completeness? It looks like completeness will also not be a topological property because the definition of completeness uses Cauchy sequences and all that and not just open sets.

However, you will see in a future video that compactness is also defined in terms of sequences. In fact, I can briefly give the definition and say a metric space is said to be compact if every sequence has a convergent subsequence, exact same definition that we saw for topological for topology sorry for the real numbers. So, compactness is also defined in terms of sequences, but compactness turns out to be a topological property.

So, just because something is defined in terms of sequences does not mean that it is impossible for that to be a topological property ok. It just so happens that you can characterize completeness entirely in terms of open sets same remarks apply for closed sets, closed set is also defined using sequences; however, you can characterize closedness completely by open sets ok.

But completeness is not a topological property, to see that what you do is consider this set 1 by n which is a subset of R ok. So, of course, n coming from the natural numbers, here you can put the usual metric that is the subspace metric ok. Now any subset of 1 by n will be open this is something that will be utterly easy to see ok.

So, what is this, what is this tell us? Well, note that 1 by n under discrete metric under discrete metric gives the same topology gives the same topology, that is if you put the discrete metric on the set 1 by n you get exactly the same open sets as when you put the Euclidean absolute value metric on 1 by n, but in an earlier exercise you would have explored and you would have realized that any set with discrete metric is actually complete.

If you have not explored that please do so, now, it is important any metric space that is coming from the discrete metric is automatically complete. So, when you put the discrete metric on 1 by n it is going to be complete; obviously, when you put the usual metric it is not complete because 0 is not there as an element of 1 by n and 1 by n converges to 0.

This shows that you can have two different metrics on the same set which give rise to exactly the same topologies, but one of the metric spaces is complete the other is not. This concludes this video on Completeness Continued and you are watching this course on Real Analysis.