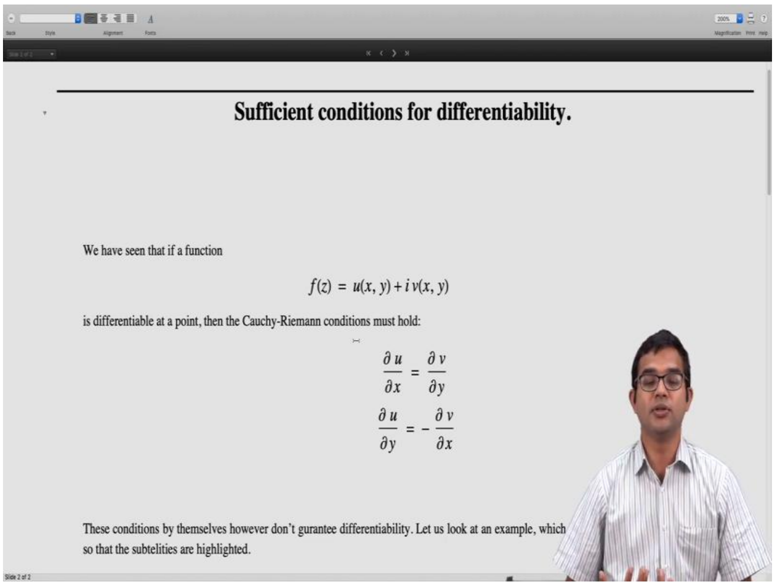
Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Complex Variables
Lecture - 09
Sufficient conditions for differentiability

So, we have looked at the Cauchy-Riemann conditions. We have seen how if a function is differentiable at a point, a function of a complex variable is differentiable at a point, then it must necessarily satisfy the Cauchy-Riemann conditions right. So, it turns out that the Cauchy-Riemann conditions are not quite enough, they are not the sufficient conditions for differentiability and so that is the topic for this lecture right.

So, we need something slightly more than Cauchy-Riemann conditions, this is a somewhat technical discussion; but we included in the interest of completeness right. So, we will look at an example of how you know these rather technical conditions may also play a part right.

(Refer Slide Time: 01:11)



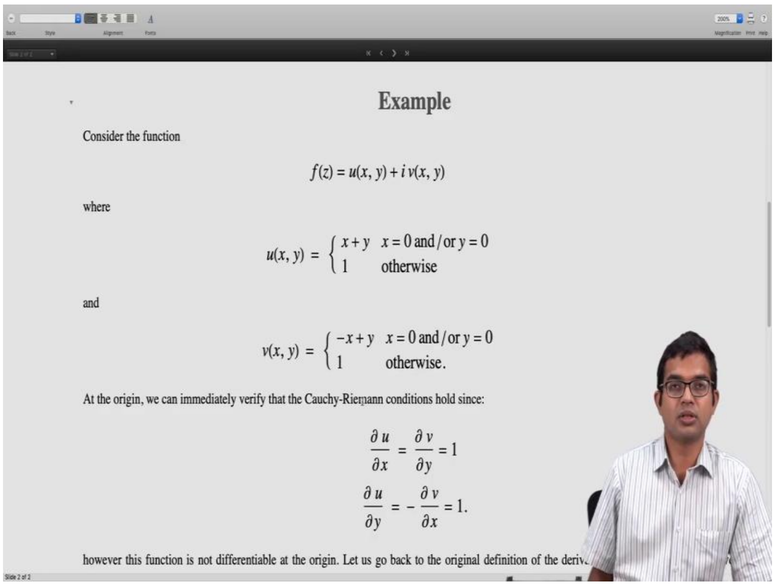
So, given some function f of z and so, it has a real part and an imaginary part and so we might think of a function of a complex variable as really you know two functions of two real variables. So, it is made up of a u and it is made up of a v. But we have seen that both u and v being very nice functions of x and y does not guarantee that the function f of z itself is a nice function of z right; meaning, differentiability of the function f of z is established only if you know not only that u and v very nice, but they are also sort of intimately connected to each other right.

So, in particular if f of z is differentiable at a point, then we have seen that the Cauchy-Riemann conditions must hold and so, dou u by dou x must be equal to dou v by dou dou y and dou u by dou y is equal to minus dou v by dou x right. So, this is a necessary condition, but these conditions by themselves do not quite guarantee differentiability. So, let us look at an example which is a rather contrived example, where the Cauchy-Riemann conditions hold; but differentiability is not guaranteed right.

So, I mean in some sense, differentiability requires that the derivative is well-defined and the derivative is well-defined if the limit which comes in when we define the derivative is well-defined and the way that the limit is well-defined when you are dealing with a function of a complex variable is if it is; if it takes the same value no matter in which direction you approach that point right.

So, that is the absolute key starting point and so, we will see how there is this artificial example, where the Cauchy-Riemann conditions hold at a point, but it is not differentiable.

(Refer Slide Time: 03:19)

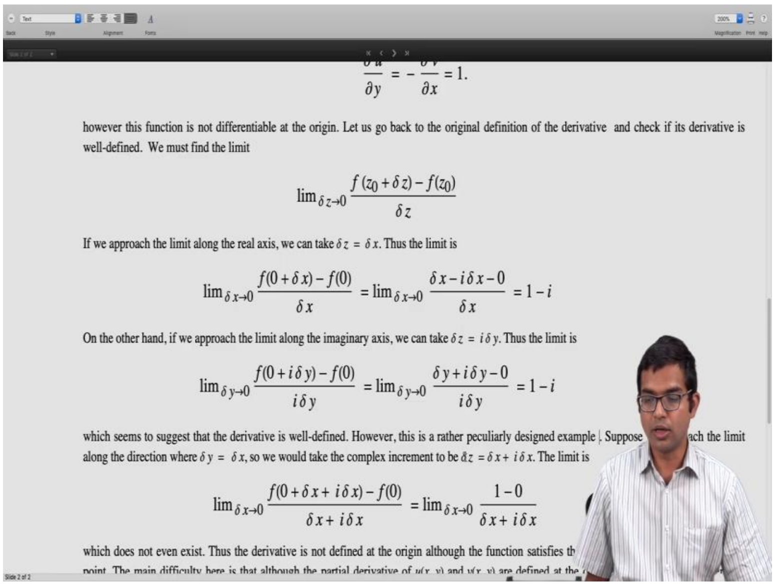


So, consider this function f of z. So, this function is defined in terms of u and v right. So, u of x,y is separately defined. It is defined to be x plus y, if you are on the x axis or on the y axis, if x equal to 0 or y equal to 0 or and x equal to 0 and y equal to that is the origin also has this x plus y.

And if everywhere else in the plane, it is just one right and likewise, v of x,y is defined as minus x plus y if you are on either of the axis, x axis or the y axis and it is 1 everywhere else. So, it is an abrupt function and so, that is where you will see that is how it bodes trouble right. So, we will see, but at the origin, we can verify that in fact the Cauchy-Riemann conditions do hold right.

So, if you look at the origin and compute dou u by dou x, it is just this function. So, it is going to be 1 and dou v by dou y is also 1. So, dou u by dou x is equal to dou v by dou y and dou u by dou y is equal to minus dou v by dou x which is also equal to 1 right. So, if you blindly just check this, I mean it appears like Cauchy-Riemann conditions hold; therefore, it looks like this function should be differentiable. But let us go back to the original definition of the derivative and check if its derivative is well-defined right.

(Refer Slide Time: 04:53)



So, how do we check this? We must work out this limit, limit of f of z0 plus delta z minus f of z0 divided by delta z and delta z goes to 0 and regardless of which direction this delta z goes to 0 along, you must get the same value. If that happens, then it is a well-defined derivative. So, let us start by approaching along the x axis. So, which means that we take delta z to be delta x and thus, the limit is simply given by limit delta x going to 0 f of 0 plus delta x minus f of 0. The whole thing must be divided by delta x.

Now, what is this function at this point 0 plus delta x? So, 0 plus delta x means it has only real part and so, and the imaginary part is 0 right. So, y is 0 according to this definition. If y is 0, then you have to use the first one. So, that gives you u of u. So, u of x, y is x plus y right. So, in this case y is 0.

So, you have delta x delta x minus i delta x. So, minus i delta x comes from v right. So, you have a plus x and a minus x. Plus x in this case, it is delta x and minus x, so you have to put i times v of x,y. So, it is minus i times delta x minus f of 0 is just 0 right. So, divided by delta x.

So, then of course, delta x cancels out and so, you are left which is 1 minus i. So, it is a very straightforward exercise. It is just using the definition to compute this limit when you are approaching you know delta x equal to 0, if you are coming along the x axis. Now, we can do the same thing along the y axis. If you are approaching the origin along the imaginary axis, then we know that we must take delta z to be i times delta y.

So, thus, the limit is limit delta y going to 0 f of 0 plus i delta y minus f of 0. The whole thing must be divided by i times delta y. After all, delta z is i times delta y and so, what about this function? So, now, as far as this function is concerned you know x is 0, but y is delta y.

So, if y is delta y, so again you know x plus y and minus x plus y must be used. So, it should be just delta y plus i delta y right. So, v is just y delta y. So, therefore, u plus i v is delta y plus i delta y minus 0 because the function at the origin is 0, the whole thing must be divided by i delta y.

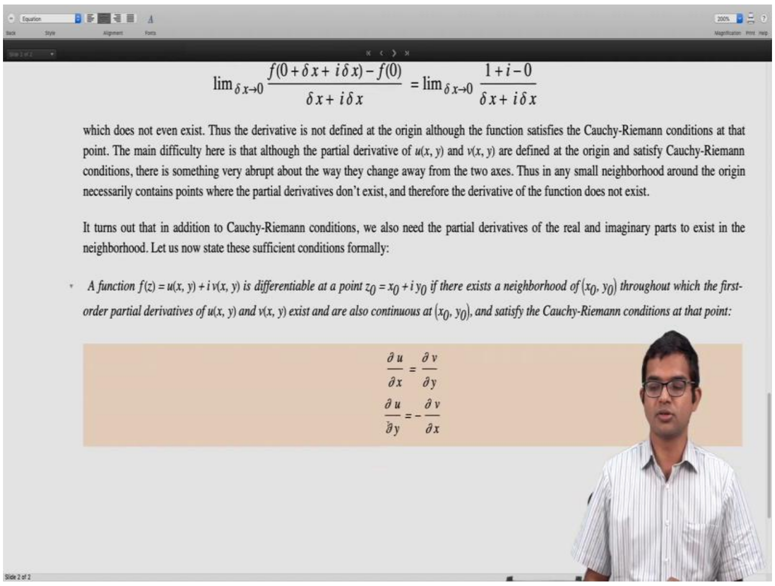
So, you get a cancellation of delta y above and below and so, you are left with 1 plus i divided by i which is 1 minus i right. You can check this. So, indeed, the limit is the same whether you approach the origin along the x axis or along the y axis. So, if we are in a rush, we might be tempted to think that you know the limit seems to be all good; it is the same, no matter which direction we approach from.

But we need to check this one more direction and then, we see that it all falls apart right. So, yeah, so like I said this is a rather peculiarly artificially designed example. So, if we approach, if we approach the limit along you know along a direction which is 45 degrees to the x and y axis. So, then you would say that delta y is equal to delta x, then we would take the complex increment to be delta d z.

So, you can also approach it along some angle theta other than 45 degrees, but I am just taking 45 degrees for convenience. So, then, you will see that the limit is delta x going to 0 f of 0 plus delta x plus i times delta, delta x. So, now, you have a real part and an imaginary part minus f of 0 divided by delta x plus i delta x. So, that is the key point.

So, if there is both the real part and imaginary part, then this function suddenly takes a value 1 right. So, since it takes since it takes the value, so the x component is delta x and the y component is also delta x and so, both u and v have to be 1. So, in fact, I should write it as 1 plus i.

(Refer Slide Time: 09:56)



So, this should be 1 plus i. It does not matter so much. So, whether you write it as 1 or 1 plus i. So, the key idea is that the numerator has just a constant. There is a constant in the numerator, but the denominator is becoming arbitrarily small. There is a delta x. So, there is nothing in the numerator to cancel out this delta x, which means that the limit does not exist.

Now, suddenly it is blowing up right. So, this limit seems to be well-defined only if you approach along the x axis or along the y axis right. So, otherwise it is not even well-defined. So, therefore, differentiability falls apart here.

So, there is no derivative of this function at this point. It is a very artificial type of a function; but it is there to illustrate that you know you may have Cauchy-Riemann conditions holding which is true here and yet the derivative is not well-defined.

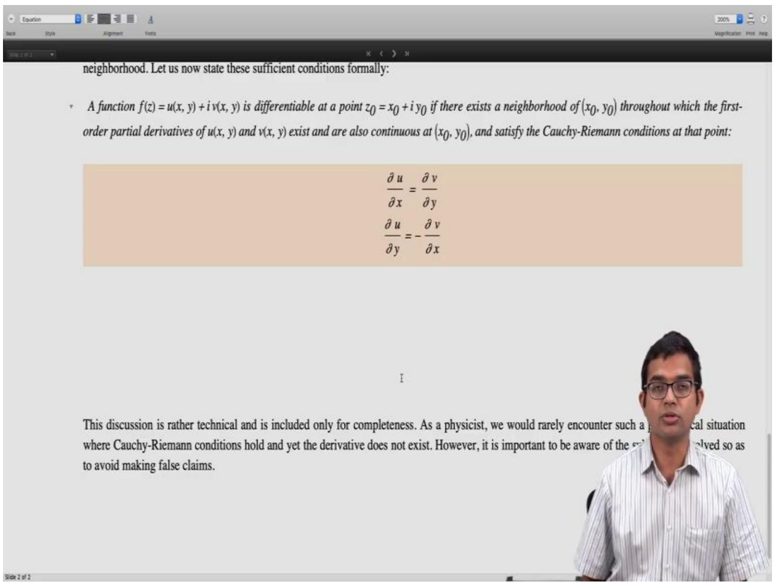
So, you know the main cause of difficulty here is you know these partial derivatives are not well-defined in the entire neighborhood of the point of interest. So, the point of interest is the origin. So, although at the point you have dou u by dou x and dou u by dou y and dou v by dou x and dou v by dou y, they are all well-defined and also they are conspiring in exactly the right way right.

So, Cauchy-Riemann conditions do hold. But it turns out that you need more than that for this function to have differentiability at that point and that requirement is that you know there is a way to prove this rigourously. Of course, we will not go into it. But, so, the statement is that if all these partial derivatives exist in the neighborhood around the point of interest.

And if there is continuity of these partial derivatives at the point of interest and Cauchy-Riemann conditions hold, then your function is going to be differentiable. So, in other words, stated formally. So, if you have a function f of z which is given as u of x,y plus i times v of x,y, if it is differentiable; it is differentiable at a point right. So, this is a sufficiency condition.

So, differentiable, it is differentiable at a point z0 equal x0 plus i times y0, if there exists a neighborhood of x0,y0 right. So, which means that you have to consider some circular region with some radius right, whose center is x0,y0 throughout which these first order partial derivatives must exist. And they must also be continuous at x0,y0 and in addition to satisfying the Cauchy-Riemann conditions at that point right. So, this extra technical requirement is also essential for differentiability right.

(Refer Slide Time: 12:51)



So, like I said at the beginning. So, this is a rather technical and you know somewhat involved idea and as a physicist this is the kind of detail that one usually does not need to bother about and is somewhat of a pathological situation. But we include this in such a discussion here for completeness and just to be aware that one has to be careful with making statements in relation to differentiability.

Cauchy-Riemann conditions for sure are essential, but something slightly more is also required for sufficiency. So, that is all for this lecture.

Thank you.