Mathematical Methods 2
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Module - 08
Partial Differential Equations
Lecture - 75
From the random walk to the diffusion equation

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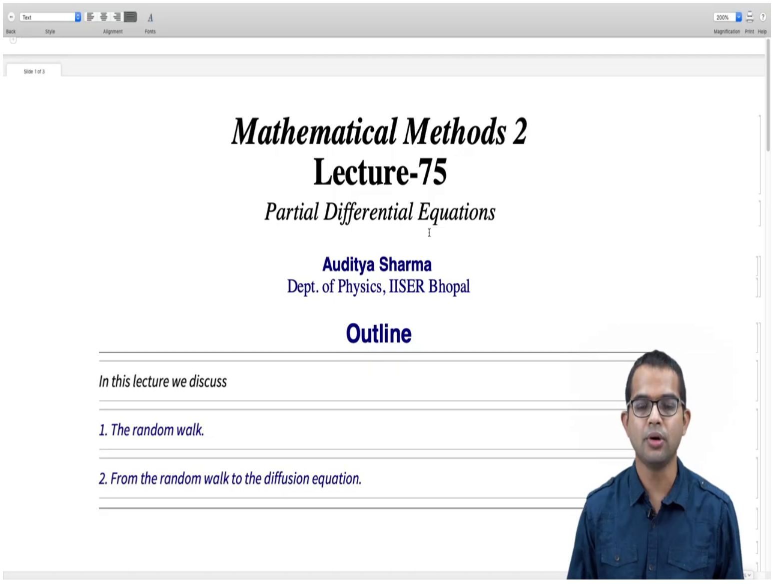

Ok. So, in this lecture, we will look at another way of arriving at the diffusion equation. So, we will look at what a random walk is. And then from this sort of microscopic picture, we will see how we can also go towards the diffusion equation by going from a discrete model to a continuous version of the discrete model ok.

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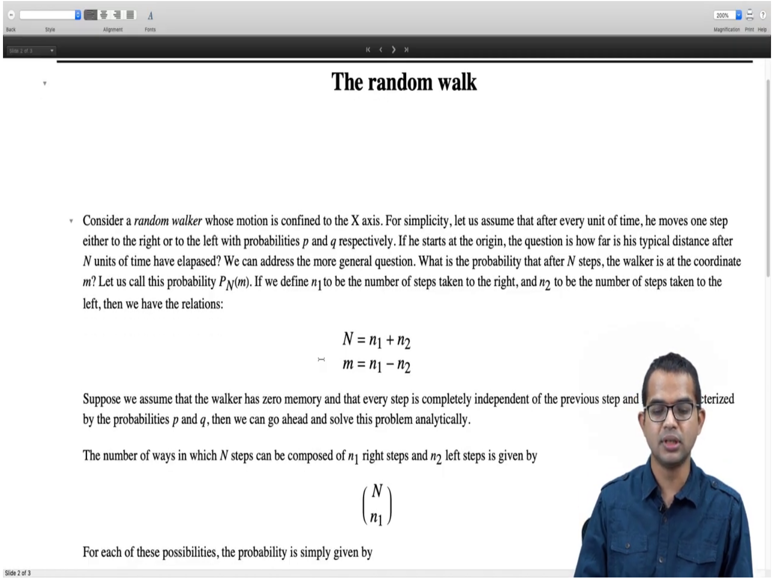

So, the random walk problem is the following. So, there is this random walker who is confined for simplicity to the X axis, you can have you know higher dimensional versions of this problem. So, you know for after every unit of time, he moves either to the right or to the left. So, with probability p, he moves to the right; and with probability q, he moves to the left.

And so there is no you know chance for him to just stay at that point you know at every instant of time, he does one of these two actions. And there is a certain unit distance that he takes right. So, after n units of time, the question is how far away from the origin has the random walker managed to go typically right. So, this is you know the type of question which is addressed in this class of problems right.

So, in general you can ask you know what is the probability that after n steps that the what is the probability that the random walker is at a particular position r right? So, this question can be asked. And in fact, there is an analytical way to answer this question. So, let us call this probability you know P N of m, so the probability that the walker is at the coordinate m after taking capital N steps.

So, if we define n 1 to be the number of steps taken to the right, and n 2 to be the number of steps taken to the left, then we clearly have the total number of steps is just n 1 plus n 2. And you know the position of this person is simply given by n 1 minus n 2 right. So, n 1 to the right and n 1 steps to the left. So, he is going to be at the point m is equal to n 1 minus n 2.

So, suppose you know there is no memory for this random walker right. So, this is what is called a Markov protocol which is that at every instant of time regardless of what the I mean history of this trajectory of this walker is going to go to the right with probability p, and to the left with probability q and nothing else. There is no dependence on history.

So, if this happens, then it is a fairly straightforward problem to solve. So, the number of ways in which you know n steps can be composed of n 1 right step, then n 2 right steps is clearly just N capital N choose n 1.

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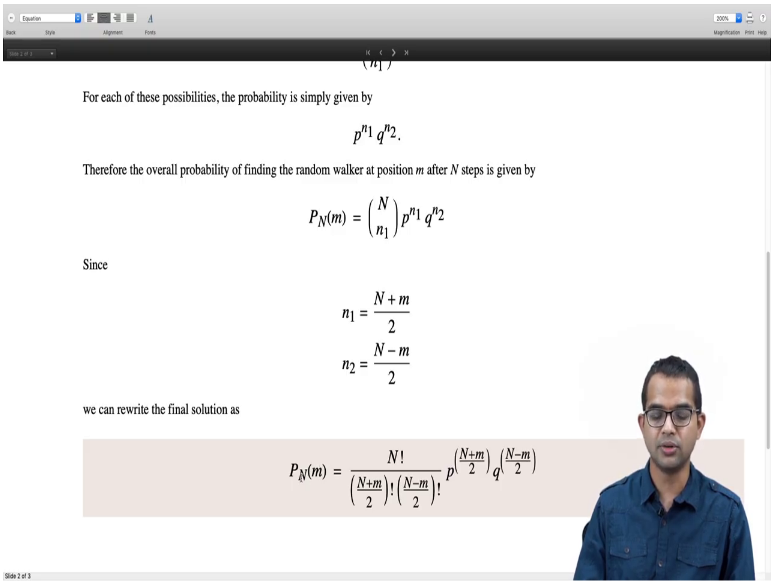

And for each of these possibilities, the probability is given by p to the n n 1 times q to the n 2 right each of these steps is an independent step. So, you have taken n 1 steps to the right. So, you get p to the n 1, and then you have to multiply with q to the n 2. So, therefore, the overall probability of finding the walker at the position m after N steps is given by N choose n 1 times p to the n 1 times q to the n 2, but we can work out this n 1 and n 2 in terms of capital N and small m.

So, it is clear that you know just from these equations n 1 is simply given by capital N plus m divided by 2, and n 2 is given by capital N minus m by the whole thing divided by 2. So, we can rewrite this probability you know P N of m as just N factorial divided by N plus m by 2 the whole factorial times N minus m by 2 the whole factorial times p to the N plus m by 2 times q to the N minus m by 2. So, clearly you know both N and m have the same parity right.

So, if the total number of steps taken N is even, so the person is going to be at an even location. And if the total number of steps taken is odd, then it is guaranteed that it is going to be at an odd location right. So, therefore, you know this is going to be an integer, this is going to be an integer. So, there are no difficulties with evaluating factorials and so on ok.

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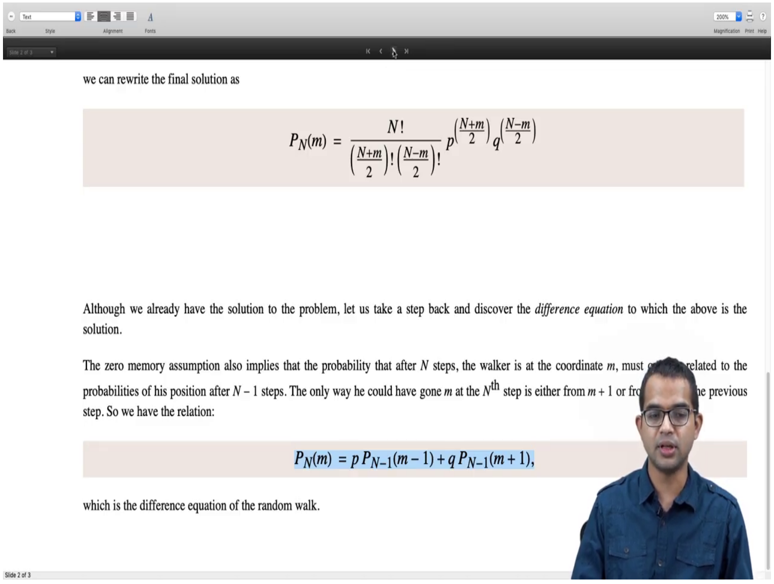

So, we can take a step back and see if we can write down a difference equation right. We have already worked out the solution to this problem, the discrete version of this problem, but it is instructed to be able to write down the difference equation to which this is a solution right.

So, the idea is that the probability that this walker is going to be at the position m after N steps is the same as saying how did he arrive at this position m right. He must have either got to m from m minus 1 or from m plus 1.

So, the penultimate location of this walker must be one of these two. Now, if he has arrived at N starting from m minus 1, then there is this probability p with which he could have moved to the right from m minus 1.

So, P N of m must be equal to p times P N minus 1 of m minus 1, or the other one other possibility is that he was initially one step earlier at N plus 1 and then he moved left. So, then you have plus q times P N minus 1 of m plus 1. So, this is the difference equation of the random walker.

Now we will see that if you take this difference equation and then make this you know continuous right. So, this is a discrete time discrete in space, but if you make it continuous in both of these dimensions, then we are going to actually get a partial differential equation which you will see is a familiar PDE.

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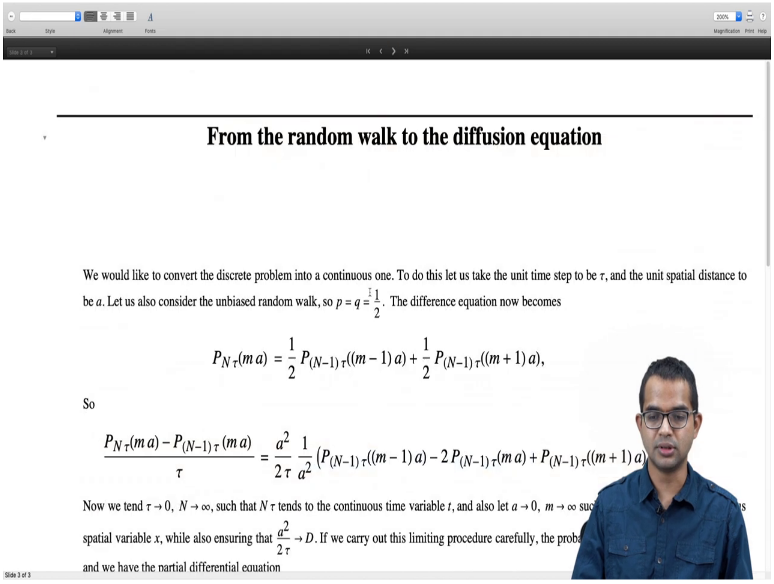

So, let us do this. Let us carry out this exercise. In order to do this, first of all let us say that it is an unbiased random walker. So, p equal to q equal to half. So, with probability half is going to go to the right, and with probability half is going to go to the left.

So, now, let us bring in a unit time step right, so to be tau, and a unit spatial distance so we are on a lattice whose lattice constant is a. And so the difference equation now can be written in this form. So, P at N times tau.

So, if you take N steps the time that has elapsed is N times tau and the position that the random walker is at is m times the distance covered. Now, this is equal to half of P N minus 1 tau of m m or m minus 1 times a plus half P N minus 1 or tau this is the time and the location is m plus 1 a right. So, now, we just rearrange this: subtract the left hand side with P N minus 1 of times tau of m times a divided by tau right.

So, if you do this and then the right hand side you see can be written in this very nice convenient form. You observe that on the left hand side you know there is m of m a, so the location is the same on the left hand side. So, it's only N tau and N minus 1 tau on the left hand side.

And on the right hand side, you see that the time at which you know this is being looked at is all these times are the same, whereas, the locations that are changing m minus 1 times a, then there is m a, and then m plus 1 a on the right hand side.

And so we are going to take this, you know take these dual limits. So, we are going to make these time steps arbitrarily small, and the number of steps taken to get to you know this position m a, the number of steps N will be made arbitrarily large, such that N times tau must tend to a continuous variable. So, although N is very large, and tau is very small, N times tau is going to give you a finite continuous variable t.

And also a is going to be made arbitrarily small, so m has to become very large, so that m a itself tends to this continuous variable spatial variable x. And while simultaneously ensuring that this a squared by 2 tau is also a finite quantity, we will identify this with the symbol D right.

So, if we carry out this limiting procedure, we observe that the left hand side is really a discrete derivative right with respect to time right. So, it is N tau, then N minus 1 tau, and then you are taking this tau going to 0 limit.

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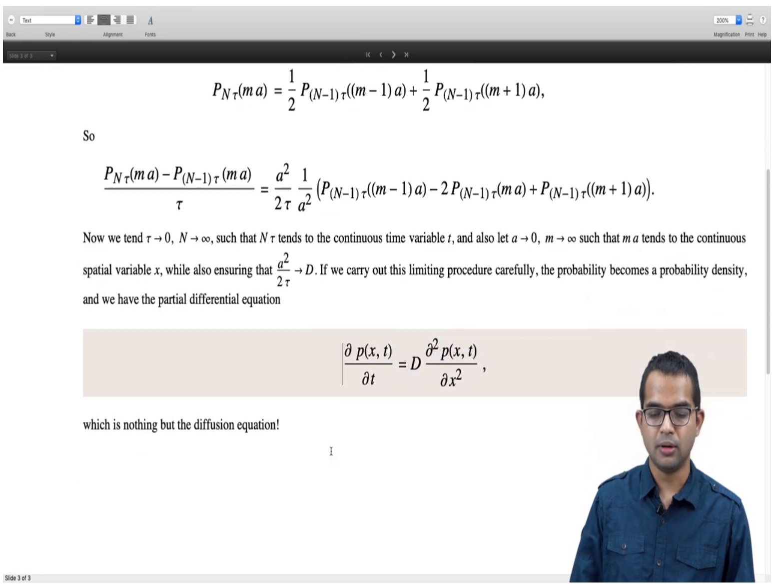

So, the left hand side actually becomes a time derivative, a partial time derivative. And the right hand side is actually a second order derivative special derivative right. So, you have this a squared sitting here. So, you can see that, so there is this forward difference and backward difference. And then you do it two times over; you can convince yourself or look up some elementary textbook where on how to take a derivative and write it as a discrete version of a derivative is exactly just this right.

And taken two times so that it should be identified as the second order spatial derivative of this function P. But there is also one more ingredient we have to put in which is that it is convenient to work with you know probability densities instead of these are probabilities when you are working in you know discrete space and time.

But then you ask you know what is the probability density associated with being at a certain point? And so the probability itself will come by considering some small special region dx, and then you multiply it, and that would give you the probability right.

So, if you look at if you change this to a probability density and then identify the left hand side as a partial derivative with respect to time and the right hand side as a second order partial derivative with respect to space, so you immediately get the diffusion equation right. So, this is a third way of arriving at the diffusion equation. And so we will solve this differential equation, the PDE in a future lecture.

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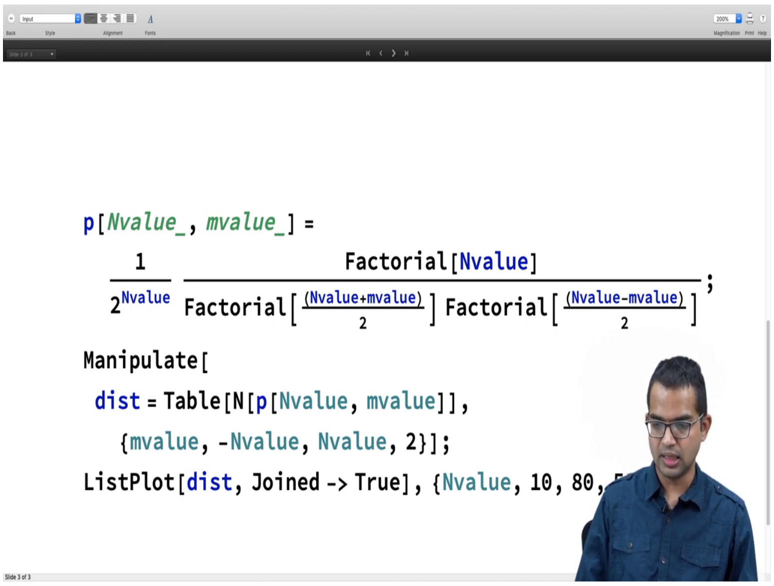

But let us quickly make a plot of the difference equation itself right. So, we have an exact solution available to us. So, let us plot this and see what it looks like. So, I have a plot of this function which is a binomial distribution really.

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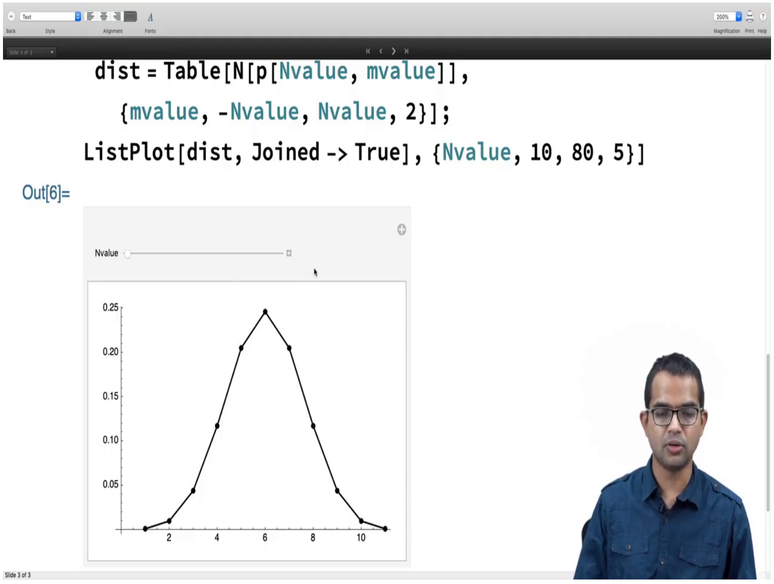

And so if you take capital N to be relatively small, it looks something like this. So, this is a distribution of you know finding your you know P N of m.

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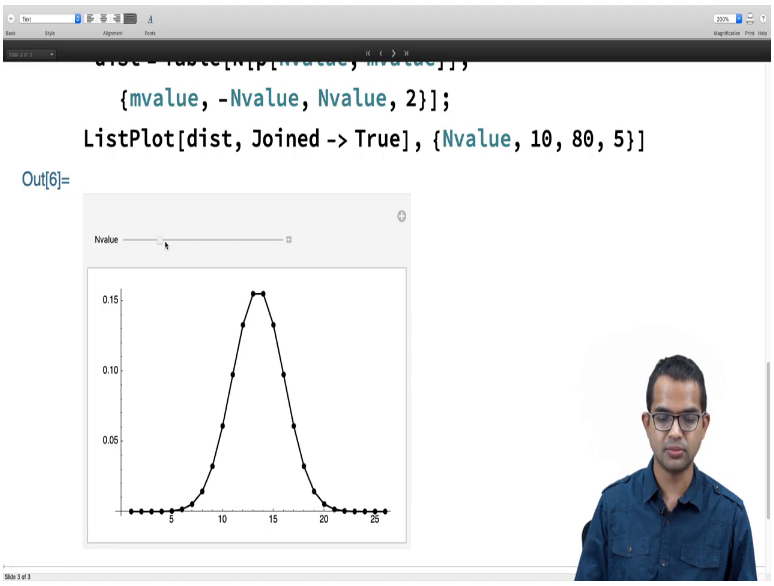

So, as you make N larger and larger, you see that this takes a form which is actually a familiar form.

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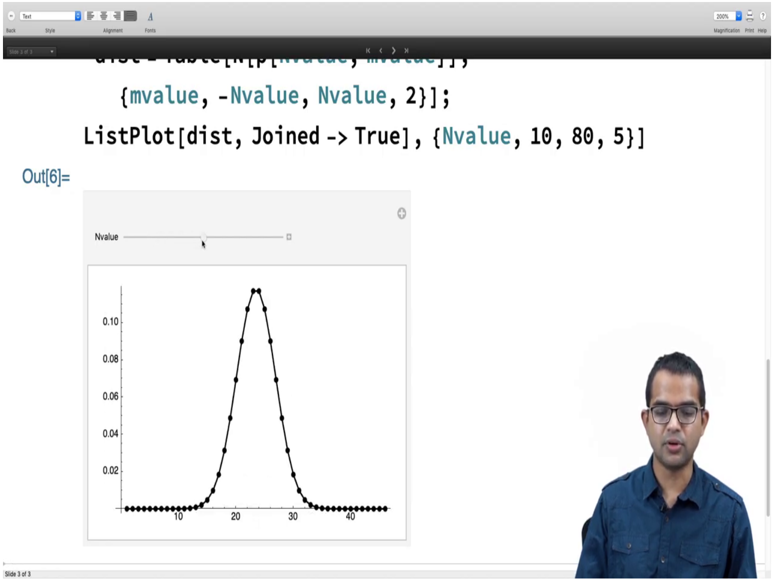

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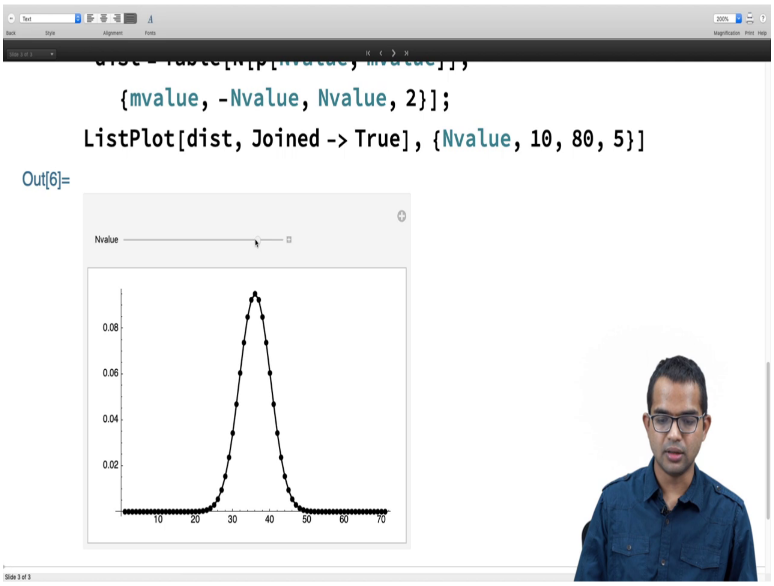

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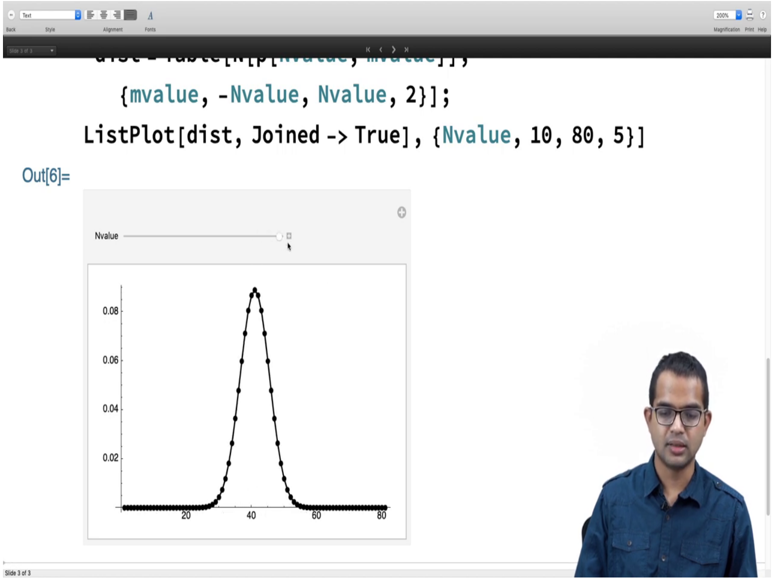

So, there is a you know, so there is this mean associated with you know this distribution, and then there is a variance associated with it right. And the shape is a familiar Gaussian. So, one thing that turns out to be typical in these kinds of diffusion equation problems is that if the random walker takes N steps, the typical position that the random walker finds himself in is of the order of square root N right.

So, although this is sort of a very generic aspect of you know the most diffusive motion right. So, we will see that. So, this is borne out in higher dimensions in other kinds of problems, and there is a remarkable generality associated with this feature. If you take N steps, the typical distance covered is of the order of square root n. So, it is useful to just look at this plot for now, but we will obtain the solution, it looks like a Gaussian.

And then we will see that when we solve the partial differential equation directly, these two will give us something which is a Gaussian solution provided and you have to supply the initial conditions, boundary conditions and so on. Those are more details. But for now, we see that even the discrete version is already pointing towards a Gaussian kind of a solution which we will work out in a you know more thorough way starting from the PDE itself ok.

Thank you.