Mathematical Methods 2
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Module - 08
Partial Differential Equations
Lecture - 71
The Laplace Equation in spherical coordinates: solution

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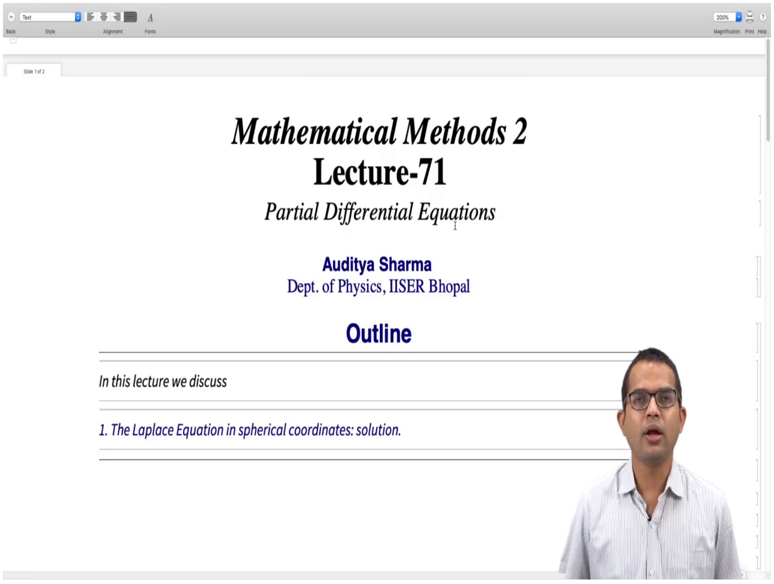

Ok. So, in this lecture we will start from the Laplace Equation in spherical coordinates and work out a formal solution for the Laplace equation when you have spherical symmetry which is inherent in the problem in terms of the boundary condition and the setup of the problem ok.

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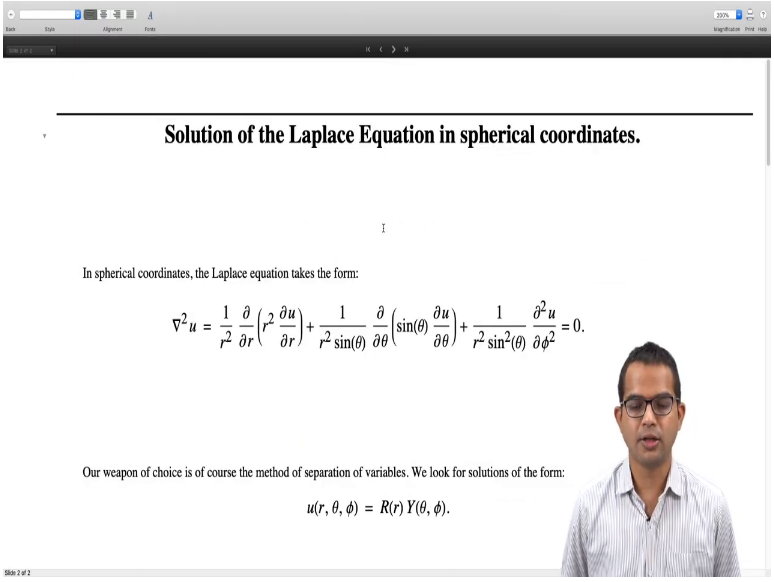

So, the starting point is the Laplace equation in spherical coordinates, we have already worked out this form. So, del squared u is really 1 over r squared dou by dou r r squared dou u by dou r plus 1 over r squared sin theta dou by dou theta of sin theta of sin theta times dou u by dou theta plus 1 over r squared sin squared phi dou squared u by dou phi squared. This must be equal to 0. That is the Laplace equation and of course, we will make use of the method of separation of variables.

We look for solutions of this form u of r comma theta comma phi is equal to r of r times Y of theta comma phi. Well, let us write it as Y of theta comma phi at this point. Then we will see that we will do a second round of separation of variables.

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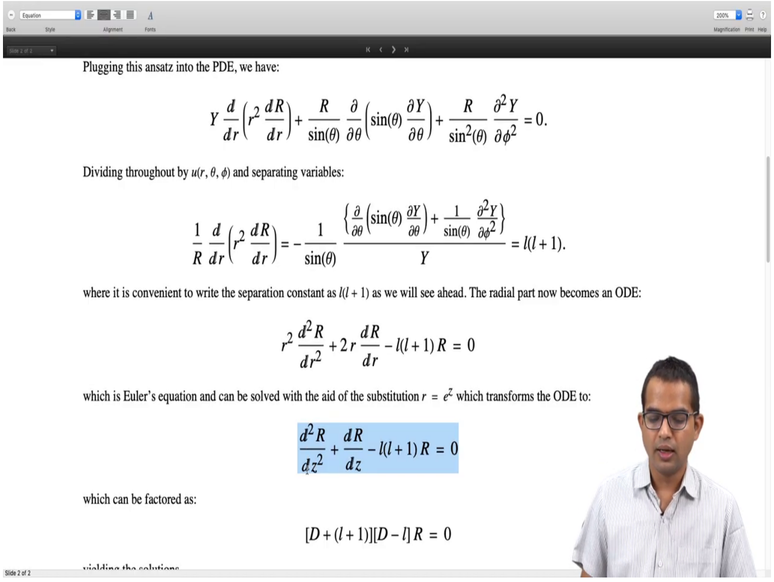

So, if you plug this you know this ansatz into the PDE. So, then we have you know Y will come out. So, then you have d by d r r squared d R by r r. So, I mean we have multiplied throughout with r squared. So, this r squared and this r squared and this r squared will go away.

So, we are left with just Y times d by d r of r squared times d capital R by dr plus R comes out again by sin theta times dou by dou theta of sin theta times dou Y by dou theta and then we have R by sin squared theta dou squared Y by dou phi squared is equal to 0.

So, now as this the standard approach we multiply through we divide throughout by u of r comma theta comma phi which is actually nothing, but this product. And then the first term, Y will go away. So, we have 1 over R times d by d r of r squared times d R by r which is equal to minus 1 over sin theta times you know 1 over Y remains here.

You know r cancels and you just have 1 over Y then you have dou by dou theta of sin theta times dou Y by dou theta plus 1 over sin theta dou squared Y by dou phi squared that is what remains in the numerator. The left-hand side is a pure function of R along and the right-hand side has no dependence on R explicit dependence, it's a function of theta and phi.

And therefore, if both of these have to be equal for all values of theta phi and r, the only way that can happen is if both of them are separately equal to a constant. And that constant its useful to write it as you know this product l times l plus 1.

We will see how this makes sense in a moment, but you know at this point it's completely general, it's a constant. And now the radial part you know becomes an ODE which is actually a familiar ODE.

So, you have r squared d squared R by d r squared plus 2 r d R by d r minus l times l plus 1 times R equal to 0. Which is actually nothing, but the Euler equation and you know you may recall from your study of ODEs, that such ODEs can be solved with you know a substitution of this kind.

In this case if you put r is equal to e to the z the ODE transforms into a you know any simpler form. So, you get d squared R by d z squared plus d R by d z you should check this minus l times l plus 1 times R equal to 0 which can be factored right. So, this is a second order differential equation which we know how to solve.

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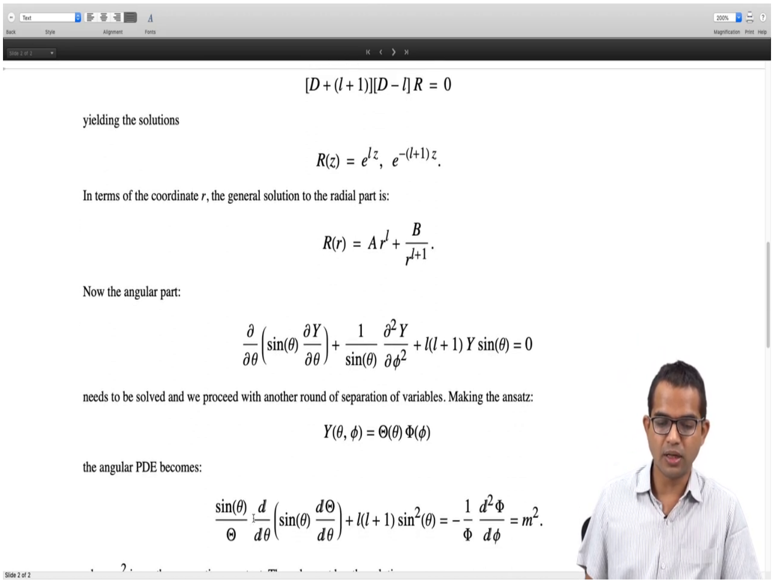

And so its straightforward to see that the solution is just nothing, but R of z is equal to e to the lz and e to the minus l plus 1 z. These are the two solutions. In terms of the original coordinates, we have you know r of r has this general solution some constant times r to the l plus some other constant divided by r to the l plus 1 right.

So, the radial part is straight forward enough. The angular part you know has both theta and phi. So, we have to do another round of separation of variables. And so we make the ansatz Y of theta comma phi is some capital theta of theta times capital phi of phi.

And so, the angular PDE now becomes you know and then we divide throughout by theta times phi and separate you know the stuff which is purely a function of theta and stuff which is purely a function of phi.

So, we get sin theta divided by capital theta times d by d theta of sin theta times d capital theta by d theta plus l into l plus 1 times sin squared of theta. You know you should check that this is equal to minus 1 over capital phi d squared phi by d d phi square.

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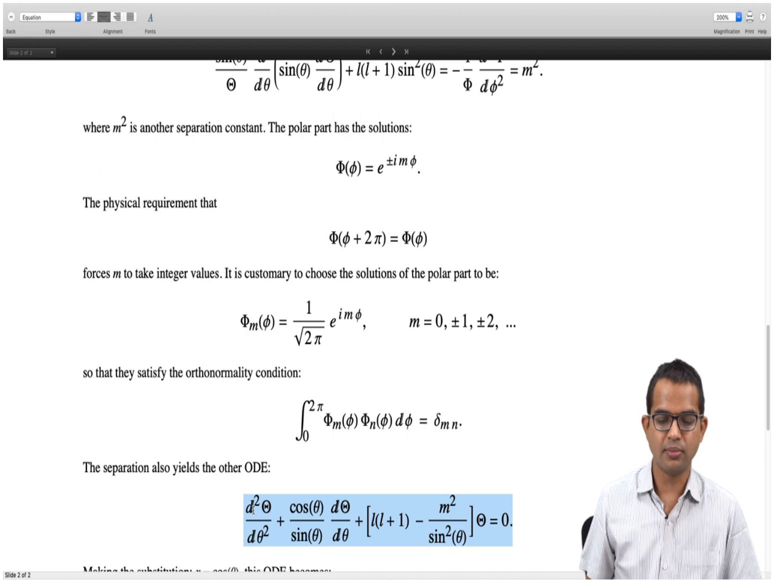

So, this is, so the right-hand side is purely a function of phi and the left-hand side is purely a function of theta. So, both of them must be equal to a constant which is conveniently put to m squared right.

So, we will see how that also makes sense. So, the polar part is readily solved right. So, this is a familiar differential equation ODE in which we know that the solutions are phi of phi is equal to e to the plus or minus i m phi. And now comes a physical requirement right. So, phi is this angle which you know which goes around.

So, you have this sphere and theta is this angle which you know which with respect to the z axis and then phi is the other angle which is the polar angle. And so the physical requirement that you know if you go around this circle and come back to where you started the solution should remain unchanged.

So, the physical requirement that this phi of you know small phi plus 2 pi when you make an addition by 2 pi there should be no change immediately forces these ms to take integer values. So, we are going to work with these solutions of phi and it's customary to choose this normalization. So, 1 over square root of 2 pi e to the i m phi, where m is allowed to take all integer values positive, negative and 0.

So, and in this form, they satisfy this orthonormality condition. So, it's useful to write down this orthonormality condition 0 to 2 pi integral phi m of phi times phi n of phi d phi is delta m n. So, the separation also yields the other ODE right. So, the other ODE is a little more complicated, but.

So, we can explicitly write down the other one in this form d squared capital theta by d theta squared plus cos theta by sin theta d capital theta by d theta plus l times l plus 1 minus m squared by sin squared theta and that should be equal to 0 right.

So, that is the other ODE which comes from just using this part along with you know m squared and you have suitably divided throughout by sin squared and that is why you have this ODE.

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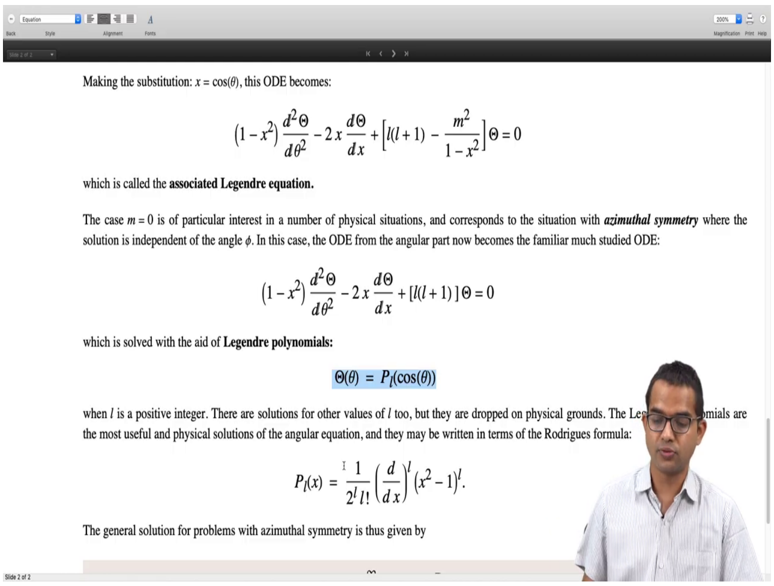

Now, this is also a well-known ODE and its form becomes even more apparent if you make the substitution x equal to cos theta and then you know work out this algebra and then you will see that this can be written in this form, which is called the associated Legendre equation right.

So, at you know the case m equal to 0 we have also studied in some detail and so that is of particular interest in lot of physical situations where you also have you know azimuthal symmetry where the solution is going to be independent of the angle phi. And so, in this case so actually when you put m equal to 0 right. So, I mean you have it's a more general ODE and you will have solutions which are you know there are.

So, well studied solutions available even for m nonzero, but the case m equal to 0 let us look at that particular case in some more detail and you have when m equal to 0 we get this differential equation which we have studied in some detail; that is the Legendre equation.

And which has polynomial solutions when l is a positive integer right. So, I mean in general even for non-integer values of l this is an ODE which can be solved, but that is going to give you unphysical solutions. We will not get into the details of what happens there, but I mean there are solutions available right. So, if you cannot find closed form solutions you can look for series solutions and so on.

But the point is that these solutions are going to be unphysical and so, we will work with you know integer positive integer values of l and which is solved with the aid of polynomials right. So, in general the solutions are not going to be polynomials, but polynomial solutions appear automatically when you choose you know l into l to be a positive integer right.

So, there is also going to be a nonpolynomial solution which again is discarded right so, but the polynomial solution is of this kind right. So, we have studied this in some detail. So, you can go back and check those earlier lectures on orthogonal polynomials. And here we will directly work with the solution right. So, this differential equation has this solution: theta is equal to P l of cos theta right. So, it's P l of x, but x is cos theta.

So, we have P l of x P l of cos theta is the solution for our theta of theta. And so we have seen that this P l of x can be written in terms of the Rodrigues formula right. So, 1 over 2 to the l l factorial the lth derivative of x squared minus 1 to the whole to the power l. So, indeed l in this form definitely must be a positive integer.

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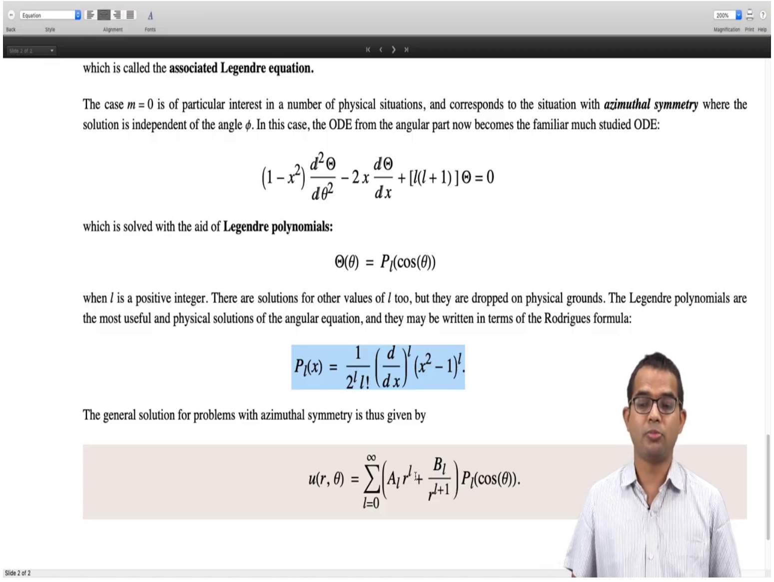

And so, the general solution therefore, for problems with azimuthal symmetry can be written down in terms of this infinite series. So, you have u of r comma theta, so there is no dependent one phi, summation over l going from 0 to infinity, you know the radial part has this both r to the l and 1 over r to the l plus 1.

So, A l r to l plus B l divided by r to the l plus 1 times Legendre polynomial of cosine of theta right. So, when confronted with problems with azimuthal symmetry, we will just directly take this as our ansatz and try to work out these coefficients.

So, right A l and B l are, of course, coefficients which have to be determined from the specifics of the problem, from the boundary conditions involved, and so we will just take this solution as a given for such problems and then work out the details of A l and B l right. So, we will look at some examples ahead, but in this lecture, we have covered the general theory for problems with spherical symmetry.

Thank you.