Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Module - 08
Partial Differential Equations
Lecture - 70
The Laplace Equation in spherical coordinates

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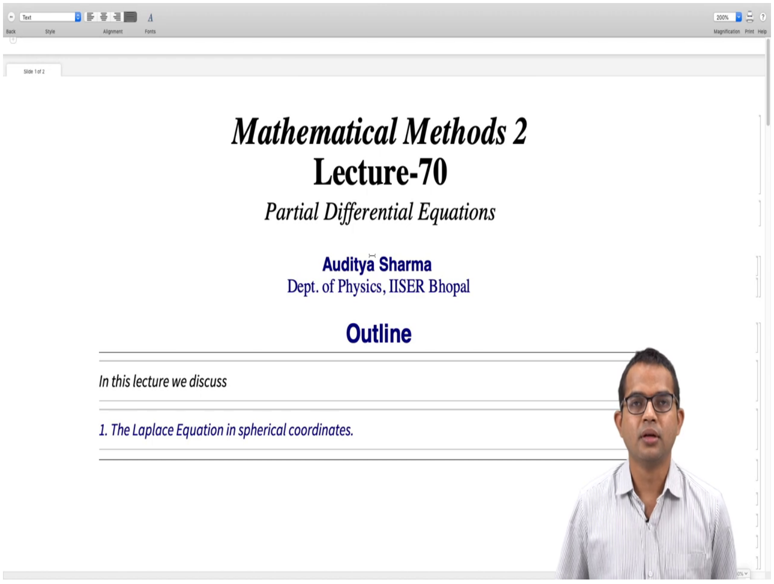

So in this lecture we will start looking at the Laplace Equation in spherical coordinates. We have seen how we can solve the Laplace equation with certain boundary conditions in rectangular Cartesian coordinates. In this lecture we will sort of set up the scene to solve the Laplace equation in spherical coordinate. So, it's kind of a review of how to you know work with spherical coordinates ok.

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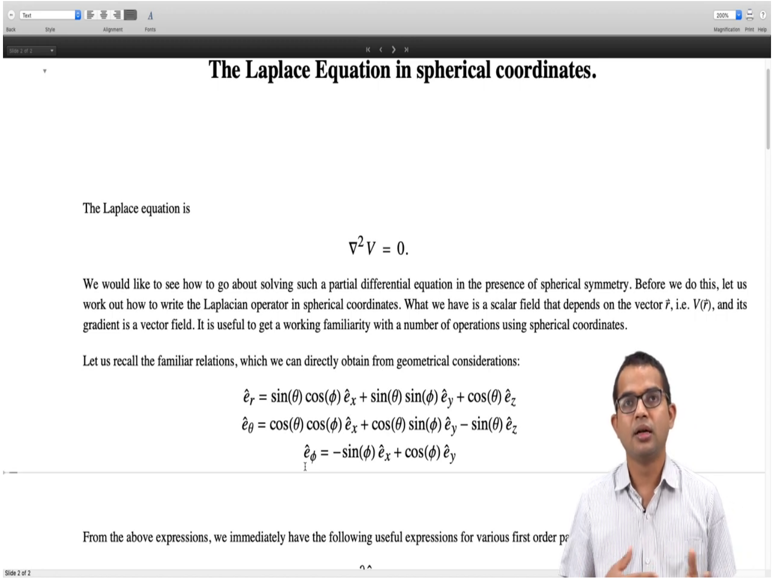

So, the Laplace equation is del squared V equal to 0. So, in this form it does not care about the coordinate system that is being used. So, we will work out how to write this del squared in spherical coordinates. There are certain problems where spherical symmetry you know exists in the problem and then it is natural to work in spherical coordinates.

So, in order to do this we will first recall that you know V is a scalar field of vector r and we will work out the gradient you know of a scalar field and then we will go towards the Laplacian right. So, you know we start with this sort of sort of familiar relations, which you know we assume that all of us are familiar with what e r is what e theta is and e phi is and just by drawing a picture directly from the geometry you know one can write down this expression.

It requires a little bit of you know thought if you are thinking about it for the first time, you must surely have seen it instead of just taking these expressions from somewhere, it is good to be able to draw a picture of a sphere and you know make an angle of theta and then imagine how this angle phi appears and then draw the small little unit vectors along the radial direction, along the theta direction tangent to the you know circle that you are traversing and also there is this phi direction which has you know another vector associated with it.

So, if you can draw this imagine it in your you know in a suitable way you can convince yourself that you know this e r is connected to e x e y and e z in this manner sin theta cos phi you know sin theta you come along, theta comes from this with respect to the z axis there is a sin theta times cos phi along e x sin theta sin phi along e y and cos theta along z that is the easiest to visualize.

And again e theta is this the vector which is you know along the tangent that you can convince yourself that its x component is cos theta cos phi, the y component is going to be cos theta sin phi and then the z component is minus sin theta and then finally, this is also straightforward to see perhaps this more the most straightforward to see is e phi is minus sin phi along the x direction and plus cos phi along the y direction.

So, all of this can be seen directly from just a good picture right. So, starting from here its useful to write down various first order partial derivatives right. So, there are multiple ways of arriving at the final result that we will be arriving having it, you can start in Cartesian coordinate write down the expression and then try to work from there and you now tedious algebra would be involved finally, will get to the same final point that we will get to in this lecture.

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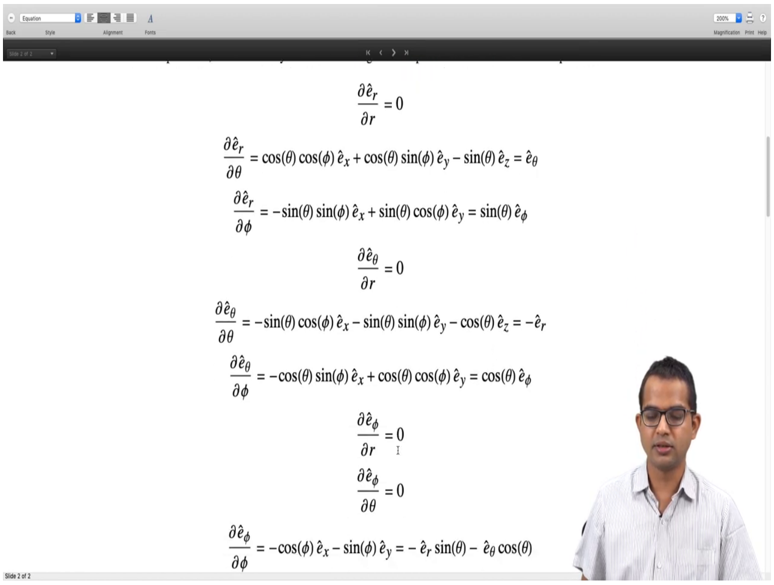

So, it's useful to see that you know this e r the direction along the radial direction does not depend on the length itself. So, as you can explicitly see from this expression there is no dependence on r. So, dou e r by dou r is equal to 0 and, but dou dou e r does depend on theta and on phi. So, it's useful to take this derivative dou e r by dou theta. This is going to be cos theta cos phi e along e x plus cos theta sin phi along e y and then minus sin theta long e z.

But then we see that this expression is actually something that we already have here and in fact, this is nothing, but e theta. So, dou e r by dou theta is seem to be just e theta and again you can check that dou e r by dou phi by taking a derivative partial derivative with respect to phi and then matching this you know the other expression you can see that indeed this is actually nothing but sin theta times e phi right.

So, that is here and then we have three such relations with respect to e theta. So, with respect to r of course, it is just 0 and then you can check directly from here from these from these relations that dou e theta by dou theta is going to be just minus e r and dou e theta by dou phi is cos theta times e phi and finally, we have these three relations with e phi. So, with respect to r if you take a derivative with 0 again here e phi does not depend on theta. So, with respect to theta also we should take the partial derivative at 0.

And then finally, you can check that if you take a derivative with respect to phi dou e phi by dou phi will turn out to be this expression you have to do a little bit of manipulation to see that indeed you can write it as minus e r times sin theta minus e theta times cos theta. So, this quantity is a linear combination of both e r and e theta which you can quickly check by using these expressions here ok.

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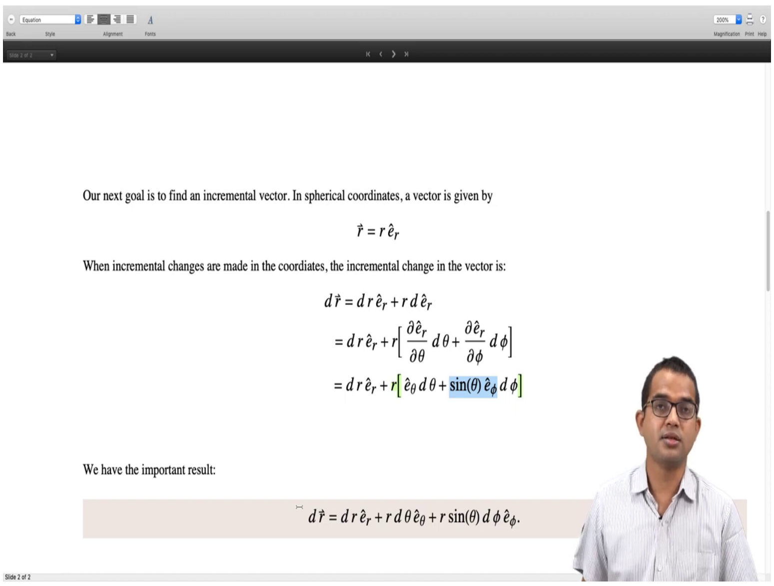

So, once we have all these first order derivatives, we are now ready to go to the next goal which is so, find the gradient of a scalar field, but before we do that let's first write down the incremental vector in spherical coordinates. So, the vector in spherical coordinates is some vector r is the magnitude of the vector times this direction and now if you want to find an incremental change in this vector, if you make some incremental changes in the three coordinates.

So, that is you know it is given by this expression d vector r is which is d r times e r along the direction of the vector itself plus r times you know change in the direction itself of this vector d unit vector e r. So, we leave it as it is and then to do the second one we use the relations which we have just obtained.

So, dou e r by dou theta d theta plus dou e r by dou phi times d phi, but this we already worked out to be e theta and this we saw is sin theta times e phi. So, together we have this important result: d r vector r is d r e r plus r d theta e theta plus r sin theta d phi along e phi right. So, this is the you know incremental change in the vector.

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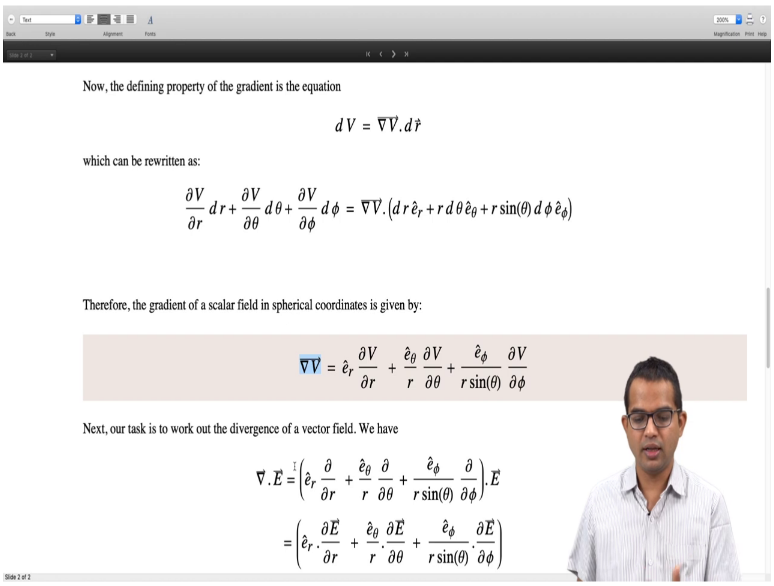

Now, using this and this defining property of the gradient. So, will actually just directly sort of write down the gradient of a scalar field in vector coordinates. So, if you make you know if you the gradient if you take the gradient of a scalar field and take the dot product with this incremental vector then it must give you the incremental change in the potential over I mean V I am assuming this potential, but the scalar field right.

So, there is a you know at every point in your space there is a scalar value that you are giving it and so, gradient is you know gives you a vector field and so, when you take a dot product with respect to d r. So, that is going to give you back this d V. So, this is like a defining property of the gradient you can go back and look up these concepts. Now so, therefore. So, dou V by dou r.

So, which yeah. So, there is another way of you know conceptualizing this incremental change in this scalar V which is dou V by dou r times d r plus dou V by dou theta times d theta plus partial derivative with respect to phi times d phi. So, this quantity is the same as this gradient V which we do not have an expression for dotted with this incremental vector which we already worked out.

So, if we write it out explicitly like here and then use the fact that the gradient of V itself has some expansion along e r and e theta and e phi. So, that expansion must be this quantity so that you will get this expression right. So, you see that if you choose your gradient vector to be you know to have these components along e r along e theta and along e phi, you can verify that indeed you get this expression which is like a you know defining relationship of this gradient of a scalar field.

Now, this relation is independent of the coordinate system you are working with and therefore using this we have managed to work out explicitly this expression for the gradient of a scalar field in spherical coordinates right. So, once we have this we can move on to our next task which is. So, from here we can actually write an expression for just you know del, del is you can think of del as an operator which acts on a scalar to give you a vector field. So, that is e r dou by dou r plus e theta by r dou by dou theta plus e phi by r sin theta dou by dou phi that is the expression for the del operator.

Now, we will take this del operator and we can take the dot product with respect to a vector right. So, then you have. So, that is the divergence of a vector field right. So, we have del dot e is going to be the same stuff you know that is the operator for del then you have to take a dot product with respect to e. So, which is some vector field e. So, that is going to give you know e r dou by dou r of e plus e theta by r dotted with dou e by dou theta plus e phi by r sin theta dotted with dou e by dou phi.

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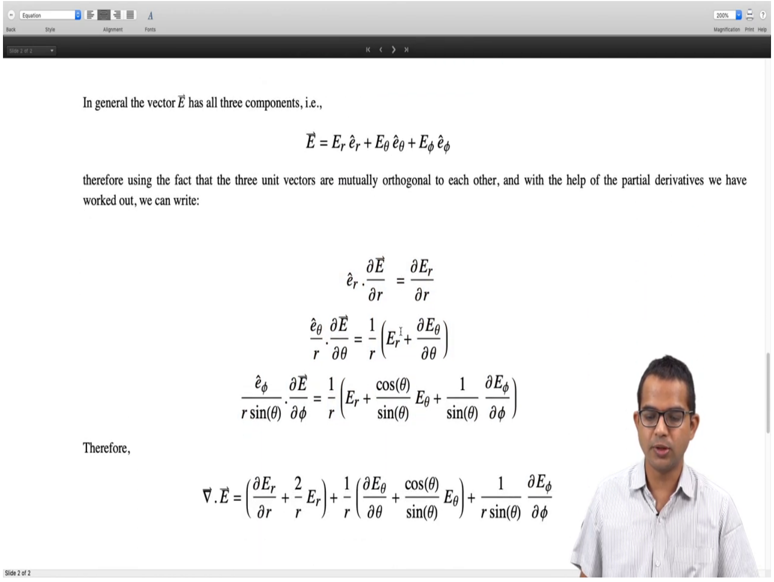

So, in general this vector e itself has three components. So, you can expand it along e r along you know e r, e theta and e phi and now you know using the fact that these are all mutually orthogonal to each other and with the help of the partial derivatives that we have worked out we can write you know. So, e r dot dou E by dou r is actually going to be just dou e r by dou r right.

So, it's only this part which is going to contribute because you are taking a dot product with e r and then when you are doing a dot product of e theta with you know with dou E by dou theta, that is going to be two terms right. So, you can check this. So, you explicitly write it down and then work out this dot product.

So, you are going to have the E r part, but there is also going to be this dou E theta by dou theta part as well will come in and when you take this dot product of e phi with respect to this, you are going to get actually 3 terms E r and then you have a cos theta by sin theta E theta plus 1 over sin theta dou E phi by dou phi right.

So, this is something that you can you know check by explicitly writing down this vector and making use of these you know all these per first order partial derivatives which we have already worked out and the fact that you know e r dot e r is 1 e r dot e theta is 0, e r dot e phi is 0 right. So, it just comes from this mutual orthogonality of these unit vector. So, this would be homework for you to check these expressions. It's very straightforward and then once you have these individual expressions we have to sum them right.

So, that is this sum, if you sum them and then group them together in a suitable way. So, we have this expression for the divergence of as vector field and that is going to be just dou E r by dou r plus 2 by r E r plus 1 over r dou E theta by dou theta plus cos theta by sin theta e theta plus 1 over r sin theta dou E phi by dou phi.

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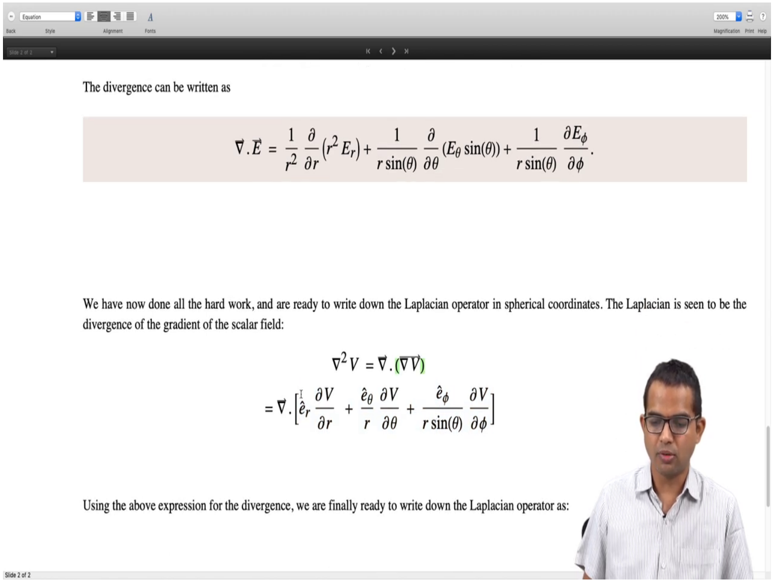

So, the divergence can be written in this compact form. So, to go from here to here you just make use of some elementary properties of you know partial derivatives. So, in fact, it's easier to go from here to here. So, you just take these derivatives and convenience yourself that term by term all these terms work out.

So, we have this, you know, important relation for the divergence of a vector field in spherical coordinates. So, you have this stuff which comes from the radial part, then from the theta part and then the phi part right. So, now, having computed the gradient and the divergence we are almost done because after all you know the Laplacian is really the Laplacian operation of del squared V is basically taking the divergence of this gradient.

So, we will make use of both the results. We have to work out the final expression which we want for the Laplacian in spherical coordinates. So, we have this divergence of the stuff which we have already have this expression for the gradient, now we have you know these components we have E r it's just like you have E r E theta and E phi we have this stuff which is going to be the component E r and this stuff along E theta and along E phi you have the components.

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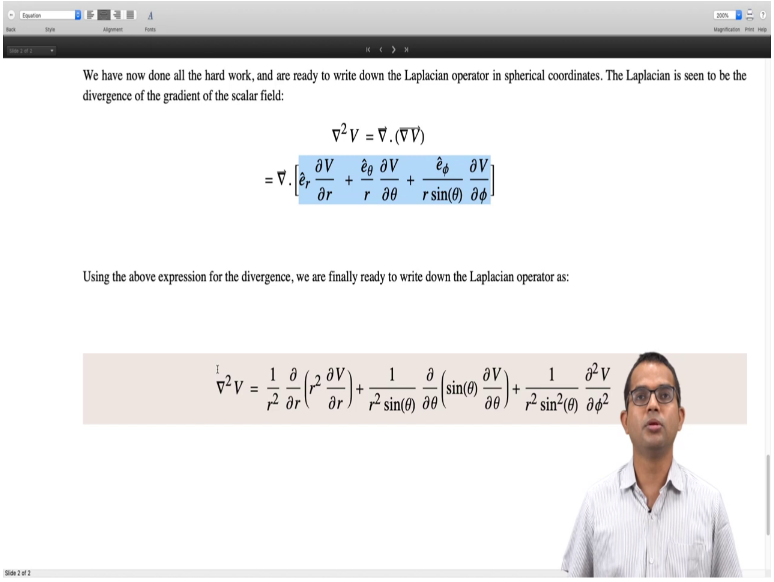

So, we just plug it into this formula and it's straightforward to see that in fact, you know there will be some small modifications in place of e r you have dou V by dou r. And then you know you know have an extra one over r squared which comes because of the terms involved here.

But you can indeed check that if you just plug in into this expression and use you know this expression here you can immediately convince yourself that del squared V is given by 1 over r squared dou by dou r of r squared dou V by dou r plus 1 over r squared sin theta dou by dou theta sin theta dou V by dou theta and once again you have an r squared here 1 plus 1 over r squared sin squared theta dou squared V by dou phi squared.

So, we will just start from this expression and make use of it to solve a problem where spherical symmetries inherent in the problem will solve the Laplace equation that is coming up later, but that is all for this lecture.

Thank you.