Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Module - 07
Partial Differential Equations
Lecture - 69
The Laplace Equation for a 3D rectangular box: An Illustrative Example

Hello everybody. In this lecture, we will look at another example of solving the Laplace equation. So, this time we will see how, so the ideas we have already explored can be extended in a natural way to three-dimensions. So, we look at a three-dimensional rectangular box and solve a Laplace equation, ok. So, again, the natural tool that we will adopt is the method of separation of variables.

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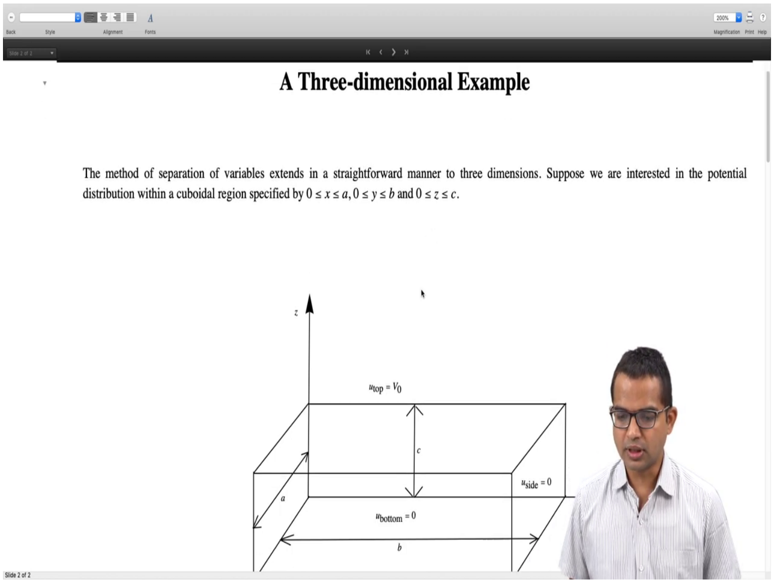

So, the problem is as follows. So, we have given this rectangular box of dynamic dimensions a, b, and c. And so, we are told that the potential on all the boundaries are specified. So, barring the top which is maintained at a potential of V naught every other side is grounded. So, the potential is 0 on all other sides.

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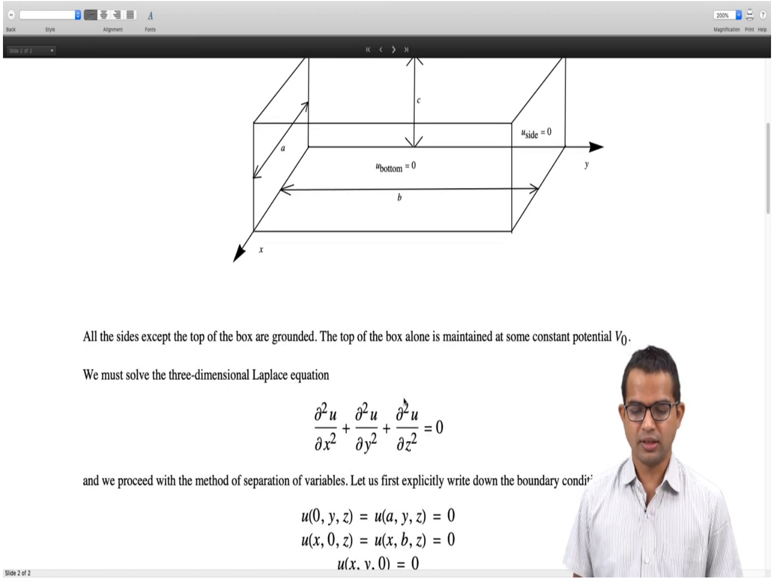

And so it is immediately seen to be a you know a version of a Dirichlet problem, right. It is a Dirichlet problem because the boundary conditions on all the surfaces are given and we are asked to find the potential within the box, right.

So, it helps to first of all write down the Laplace equation in 3D which is just this: dou squared u by dou x squared plus dou squared u by dou y squared plus dou squared u by dou z squared equal to 0. And also, it helps to write down explicitly the boundary conditions that are available to us, alright.

So, when x equal to 0 and for all values that y and z take, and when x equal to a, for all the values of y and z, the boundary condition tells us that the potential is 0. Again, when y is taken to be 0 or when y is taken to be b, so we have u of x comma 0 comma z and u of x comma b comma z, both of them separately equal to 0, we are also given that u of x comma y comma z equal to 0 is also 0. And then, the final boundary condition is u of x comma y comma c, when z equal to c the potential is V naught, right.

So, it is good to specify the problem that is given to us in this language and then make the ansatz. So, the ansatz is u of x comma y comma z is written as X of x times Y of y times Z of z now, right. So, you will see that we have to do two rounds of this usual trick of separation of variables.

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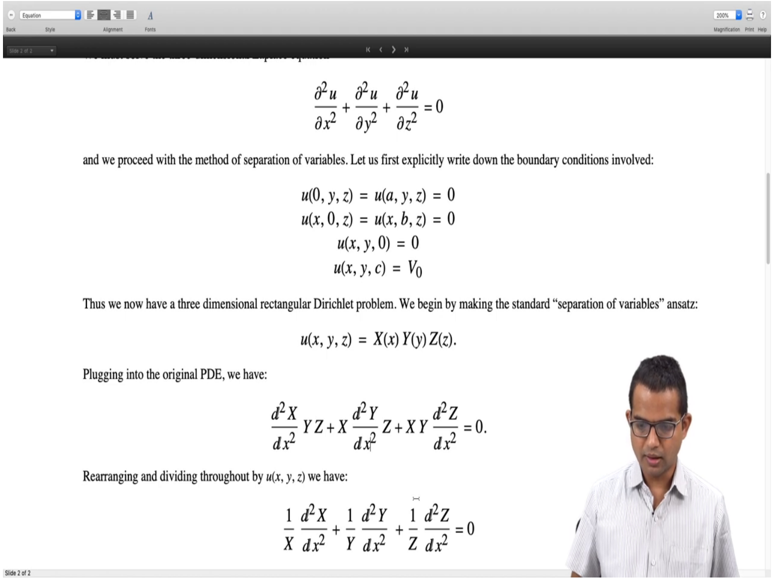

So, we first plug this ansatz into the original PDE. And so, I mean we are looking for solutions of this kind, but basically the idea is that we will stitch together solutions of this kind there is a way to do this in such a way that the boundary conditions are satisfied and there is this theorem which says that if you have a Dirichlet problem. And if you can find one solution then that is the solution, right.

So, there is a uniqueness theorem associated with the Dirichlet boundary conditions which we did not go into the details of, but it is true that if you can somehow find a solution then that is the solution. And we will see how with you know finding solutions which are separable, and stitching them together in a suitable way, we can work out the solution for this particular problem.

So, we plug it into the original PDE. So, then we get it in this form d squared X by dx squared YZ plus 2 similar terms involving the other coordinates. And then, you divide throughout by you know the product X times Y times Z, and then you see that the first term becomes 1 over X times d squared X by dx squared and the second term is 1 over Y times d squared Y by dy squared it should be, ok so, yeah.

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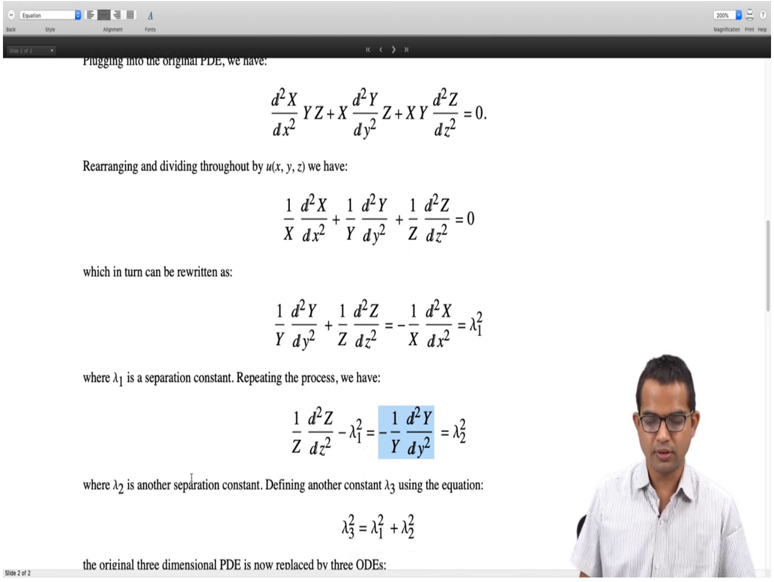

So, this is, so this should be d squared Y by dy squared and this should be d squared Z by dz squared. And so, now, so indeed and here as well. So, y and this is z squared. So, we couple these two terms together. So, this is z and so, this is y. So, it is useful to combine just these two terms together. So, we have 1 over Y plus d squared Y by dy squared plus 1 over Z d squared Z by dz squared is equal to minus 1 over X times d squared X by dx squared.

So, now comes the key argument which goes into this method of separation of variables. So, the idea is that the left hand side contains two terms which are independently functions of y and z. But the right hand side is purely a function of x.

So, you have something which is purely a function of y and z equal to something which is a pure function of x alone, for all values of x, y, and z. Therefore, the only way that this can happen is if it is a constant and that constant which we take it to be lambda 1 squared, right. Without loss of generality, you will see in a moment why for the particular boundary conditions that we are working with. So, this will turn out to be a suitable choice.

Now, if we repeat this process, there is one more round of this separation which we must do. So, then we have d squared, so 1 over Z d squared Z by dz squared minus lambda 1 squared is equal to d squared Y by dy squared. And now, once again we argue that the left hand side is a pure function of z alone, the right hand side is a pure function of y alone. Therefore, each of them separately must be equal to constant, and we will call this constant as lambda 2 squared, right. You will see in a moment why this all makes sense.

So, now, we have this left hand side is equal to lambda 2 squared minus 1 over Y times d squared Y by dy squared is lambda 2 squared. And so, there is another separation constant we have introduced, but it is useful to define another quantity which we will call lambda 3 which depends on lambda 1 and lambda 2 in this manner. So, lambda 3 squared is equal to lambda 1 squared plus lambda 2 squared. So, lambda 3 is defined using this equation.

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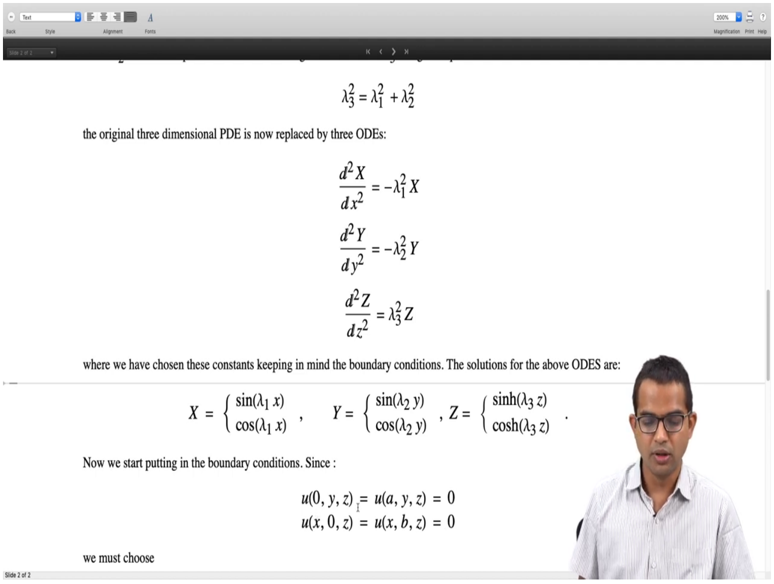

So, now the original PDE is rewritten in terms of 3 ODEs, right. So, we have d squared X by dx squared equal to minus lambda 1 X squared that is the first ODE. Then, we have d squared Y by dy squared is equal to minus lambda 2 squared times Y, and then we have d squared Z by dz squared is equal to lambda 3 plus lambda 3 squared Z, right.

So, these are familiar and simple ODEs which we know how to solve. So, that is going to give us you know sines and cosines or hyperbolic sines and hyperbolic cosines, for, in the third equation. So, let us just write down the solutions in this manner.

So, X is equal to sine of lambda 1 x or cosine of lambda 1 x, and then we will have to stitch them together in such a way that the boundary conditions work out. And then for Z alone we have hyperbolic sine and hyperbolic cosine, we could have also used exponentials, it does not matter. So, now, we start putting in the boundary conditions. So, since when x equal to 0 and when x equal to a, you must get 0 and when y equal to 0 and when y equal to b you must get 0, we must choose only sines.

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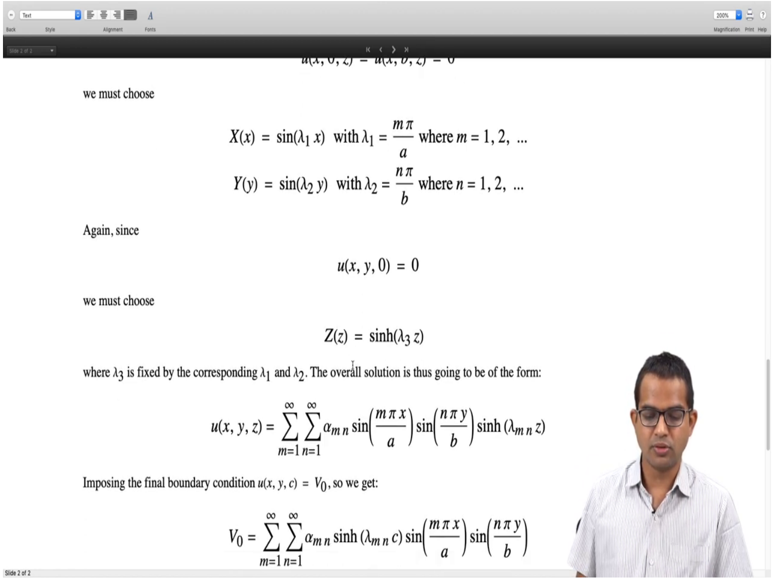

So, the cosines will not contribute because at x equal to 0 and at y equal to 0, cosine will not work out. So, the potential has to be 0. And again, the boundary condition at x equal to a and y equal to b, will force that these lambda 1 and lambda 2 must be integral multiples of pi by a and pi by b respectively, right.

So, this is like we have seen in the 2D version which is just that you have to do it you know there is one more step involved. So, you know these m's can take all these integer values 1, 2, 3, and so on and likewise with n.

Then, we have one more boundary condition which is when z equal to 0, the potential must be 0. So, we choose the hyperbolic sine, right, for z; the hyperbolic cosine will not contribute. And then of course, lambda 3 itself is connected to lambda 1 and lambda 2. If you fix lambda 1 and lambda 2, lambda 3 is automatically fixed. So, the overall solution turns out to be this double infinite series.

So, m going from 1 to infinity, n going from 1 to infinity there are these undetermined coefficients: alpha m n, sin of m pi x by a times sin of n pi y by b, then hyperbolic sin of lambda m n z. Lambda m n is of course, determined in terms of this m pi by a and n pi by b, according to you know this relation between lambda 3, and lambda 1, and lambda 2. So, there is this final boundary condition which we must impose and solve for these unknown coefficients alpha m n.

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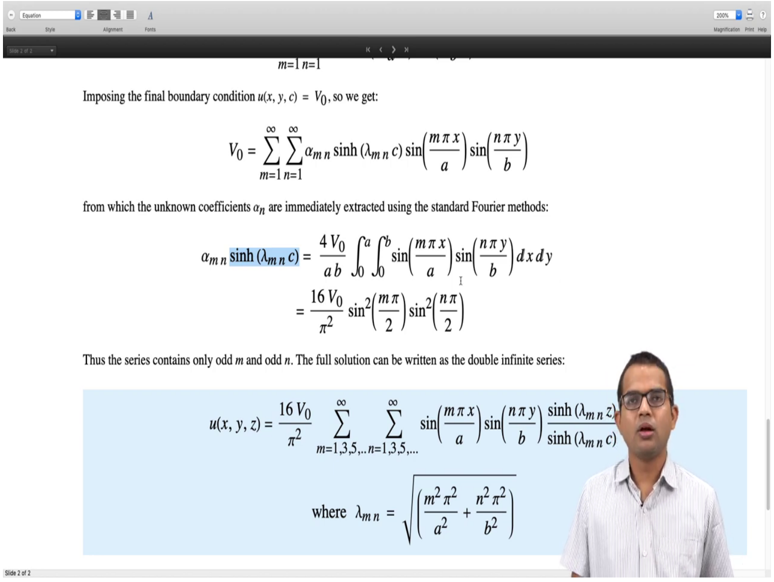

So, it turns out that this is really a Fourier series expansion problem, but there it is like a double sum involved. So, there will be double integrals to compute. So, basically, we argue that you know there is this function which is a constant which is expanded in terms of sines you know along x and y. So, we can treat this entire thing.

So, when you put z equal to c, so basically you treat this alpha m n times hyperbolic sin of this stuff to be the coefficient and that coefficient we know from standard Fourier methods you know you can check this explicitly by doing this double integral. You know you will be able to isolate the coefficient of your choice every other term will go to 0, if you do the sin of m pi x by a times sin of m pi y by b, you take an integral over dx and dy.

You know the limits being from 0 to a and 0 to b, respectively, and then you get this integral. Each of these integrals is really the same and you will get 16. So, you will get a 4 times 4 times 4, there is a pi squared in the denominator. So, the answer turns out to be quite straight forward. Only seen to be just this, and only for you know odd m and odd n. So, as sin of pi by 2 is 1, sin of pi by 3 pi by 2 the whole squared again is 1, but even integers m and even integers n are not allowed. So, the full solution is seen to be this double infinite series, where you have summation over m, m takes values 1, 3, 5, and so on; n also takes values 1, 3, 5, and so on.

And then, now you also have this sin of m pi by x sin n pi y by b times, there is this hyperbolic sin of lambda m n z divided by you know this is a constant, right. So, this came about because this coefficient had this stuff. So, that is hyperbolic sine of lambda m n c. And indeed lambda m n is of course, written in terms of m squared pi squared by a squared plus n squared pi squared by b squared the whole thing which is a square root, ok.

Thank you.