Mathematical Methods 2
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Module - 07
Partial Differential Equations
Lecture - 67
The Laplace Equation: Dirichlet and Neumann Boundary Conditions

So, we have been describing the Laplace equation and a technique to solve the Laplace equation, namely the method of separation of variables using, you know, two independent variables and using rectangular coordinates. So, in this lecture we will look at two important kinds of boundary conditions.

Now, we have this general method and solution also from within this method, and we will see how to apply it to specific boundary conditions you know taking the you know two sort of standard boundary conditions which are of importance in the problems of this kind. So, these are called Dirichlet and Neumann boundary conditions, ok.

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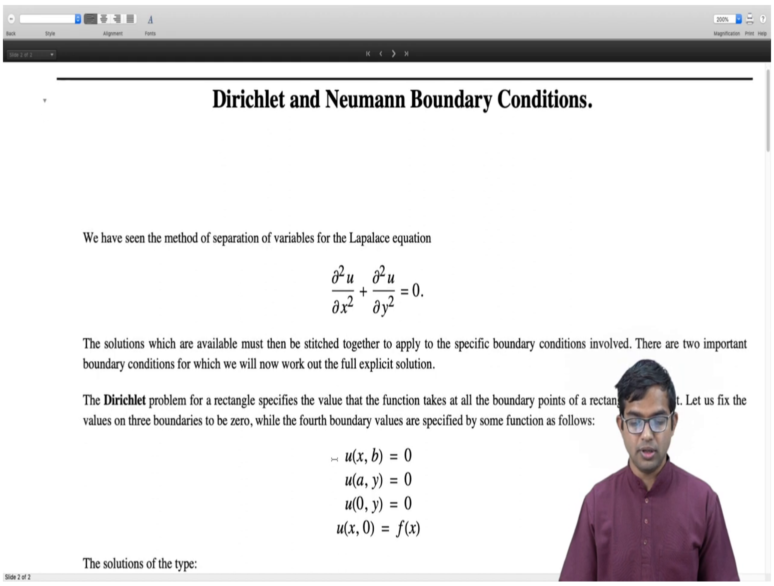

So, the Laplace equation is del squared u equal to 0. So, we have dou squared u by dou x squared plus dou squared u by dou y squared equal to 0 in cartesian coordinates and with two variables. So, we have seen how to you know write this u of x, y as some x of x times y of y.

And then you know work out the ODEs connected to capital X of x and capital Y of y and how you know we may get you know sines in one of these variables and exponentials in the other or vice versa or there is this in between case when you know if the separation constant turns out to be 0, then you can get sort of linear solutions in each of x and y.

Now, which of these do we want for a particular scenario will depend on the boundary condition. So, there are these two important boundary conditions which are the standard sort of starting points to apply the separation of variables technique and so, one of them is called the Dirichlet boundary condition.

So, the Dirichlet problem for a rectangle is where we specify the value of the function at all points on the boundary, right. So, you have a rectangular curve and so, we are interested in finding. For example, if you are interested in finding the potential you know electrostatic potential inside a rectangular region and we are given the value of the potential at every point on the boundary right.

So, and then our job would be to solve for the Laplace equation within the rectangle and ensure that you know your phi of x, y right will take the particular values which have been specified on the boundary. Now even within this very specific way of specifying the boundary conditions, it is convenient to actually set three of these to be 0 and you know come up with a function for one of these sides, right.

So, I mean I suppose we have you know in the general problem is of course you will have different functions describing the values on each of these four boundaries. So, we can actually think of this as a scenario where you add you know four different problems, all of these are sort of linear problems. Therefore, you can take solutions for one of them, take a solution for another problem or third one or fourth one and so on. And just simply add them, stitch them together literally.

You can add the solution in this case where you know in one problem you take some three different sides to be 0, but the fourth one is non-zero. You put the value of the potential at that point and you know you consider another situation, where you put a 0 in that one, but take some function for another side and then a third one where you put in the function corresponding to the third side alone.

And a fourth one where you put in a function corresponding to that side alone whereas, you keep the others 0 and then using the method that we are going to describe here, you can separately work out the solution for each of these and just simply add them all up and that will be the solution for the overall problem with the full boundary conditions.

So, it is for that reason that it is enough to study this particular case. So, the particular case I am considering is to have this rectangle which is whose two of whose sides are the x, the y axis and the x axis, but also there is a you know boundary which is set at x is equal to a, x is equal to a and y is equal to b, right. So, it turns out that at y equal to b, the value of this u function u is taken to be 0.

And on the line y is equal to x is equal to a also u of a comma y is 0, u of 0 comma y that is along the y axis is also 0. It is only along the x axis right at the bottom that the value of this function is chosen to be some f of x. I mean if this were also 0, then of course the solution would be completely trivial is 0 everywhere, right. So, some non-trivial function f of x is assumed for the bottom boundary.

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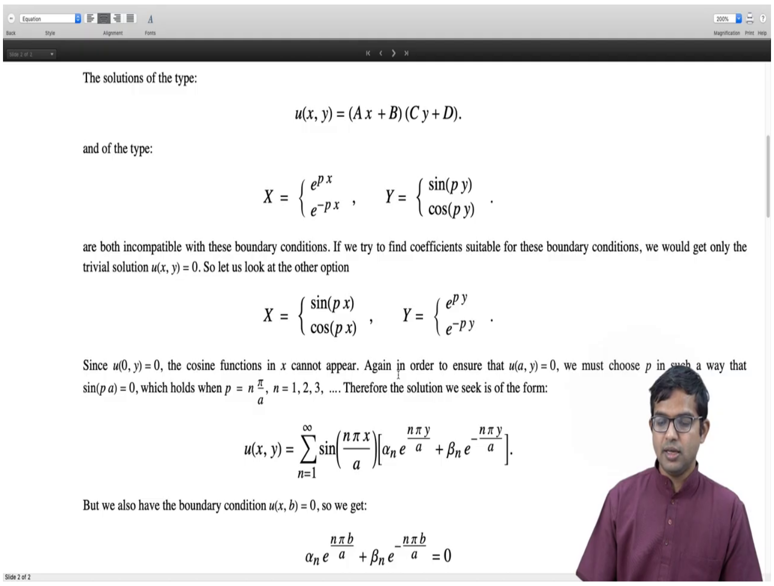

So, we will see how to solve this using the general solution that we had. So, one observation we make is that you know there are these of the three different cases of solutions possible. Actually two of them are not applicable here. So, this is something that we can sort of read off, right.

One is suppose we try to look for a solution of this linear type ax plus b times c y plus D, right. So, if you try to put it, you know when x equal to 0 x. So, suppose I put x equal to 0 and y, so I will have B into C y plus D must be 0 and so that will you know imply either B 0 or C y plus D 0. C y plus D cannot be 0 because y takes various different values.

And similarly the other conditions you can check each of the conditions if you put in basically you will end up with a scenario, where all A B C and D have to be 0. So, you get only a trivial solution. And likewise for the boundary conditions which are given here you know if you take exponentials along the x direction and sinusoids along the y, it is not going to work out right.

So, if you have to choose you know e to the p x, you know you have to choose some solution like c one times e to the px plus c two times e to the minus px the whole thing multiplied by c three times sine of py plus c four times cosine of py. Suppose you make an ansatz like this or it is not really an ansatz. Suppose you choose this to be a solution, can you fix these c 1 c 2 c 3 and c 4, such that these boundary conditions will hold, right.

So, if you put x equal to x equal to 0 and leave y as it is it has to go to 0 and again if you put a x equal to a and y leave y as it is again it has to go to 0. So, you can convince yourself that it is not going to be possible to use exponentials and also satisfy you know 0 above and a non-trivial function at the bottom, right.

So, this is something that I will allow you to play with and convince yourself. It is just maybe a few lines of algebra right to see for yourself, but basically the idea is that you try to plug in this c 1 c 2 you know put x equal to 0 and y leave it as it is x equal to a, leave y as it is and ask what conditions will it yield on c 1 and c 2 and c 3 and c 4. Put in all these four conditions and you will see that you will end up with c 1 equal to 0, c 2 equal to 0, c 3 equal to 0, c 4 equal to 0.

So, the only way you can satisfy these boundary conditions for you know if you assume this set of solutions is if you get a trivial solution which is not really useful, right. So, we want a non-trivial solution after all. I mean you also have to satisfy you know u of x, 0 equal to some function. So, the only option that can also do that and satisfy these other boundary conditions in a non-trivial way is if you choose the other possibility.

If you take the sinusoids along x, right I mean you can set them up in such a way that it goes to 0 here and it goes to 0 here and then, the exponentials can be chosen in such a way that you know at the bottom part you get a non-zero value, but only at one end you have to make it go to 0, right.

So, you can set up these coefficients. You have to take linear combinations of the exponential. So, at one end you can take it to 0, but you cannot take it to 0 at both ends whereas, with sinusoids and cosines we know that sinusoids are functions which repeatedly go back to 0.

So, you can choose your you know the other end you fix one end to be 0 and then you have to choose your other end to be such that it is at one of those one of the other 0s of sin phi. So, you will in fact get infinitely many possibilities, right which is not the case if you choose exponential.

So, basically the idea is that given the boundary conditions you must work out whether it is sinusoids which are going to be you know the right kind or if it is you know which direction do you want sinusoids and in which direction do you want exponentials, right. So, this is the first thing to work out. Once we have this, then we will see what restrictions on this p are going to be brought in by the boundary conditions, right. So, p of course at this point can be any real number right.

So, we will take you know now we will take this to be our solution and see if we can fit it to these boundary conditions, right. So, let us start with the fact that u of 0, y right. So, this is the third condition u of 0, y must be equal to 0 right. So, I mean this has nothing to do with y. It is all about x. So, if I put x equal to 0, then this must go to 0.

So, we cannot have any of these cosine functions because cosine functions will give you some stuff times y which will remain, but we do not want that. So, no matter what y is, if you put x equal to 0, it must go to 0. So, we are left with only sinusoidal functions. So, in order to ensure u of a, y equal to 0.

So, this is like what I was saying sometime ago which is that if you want you know your function to go to 0 on opposite ends, you must take sinusoids or cosines in that direction and if you want your function to go to 0 on only one end, but remain non-zero at the other end, then exponentials are appropriate right.

So, now you see that in order to ensure that u of a, y equal to 0, we must choose p such that you get back a 0 of this sinusoid. So, and that means sin of p a must be equal to 0. So, p must be some n times pi by a, right. So, a p a must be equal to some n pi. So, which is the same as saying p should be some integral multiple of pi by a right, n equal to 0 is just trivial. So, we will start with n equal to 1, n equal to 1 2 3 and so on.

Therefore, the solution we seek is going to be of this form u of x, y is going to be in general as a summation over n running from one all the way up to infinity sin of n pi x by x. Already we have got a restriction on what p can be. So, it is going to be n pi by a times x, then we have these two different possibilities. We have both this positive exponential and exponential of a negative p times y, both of these we have to keep track of.

So, there are these coefficients that will have some index n alpha n and beta n e to the n pi by y by a plus beta n times e to the minus n pi y by a. So, now we must use the other two boundary conditions to work out these coefficients. Now, one of these conditions is going to actually connect this alpha n and beta n. So, if we take the boundary condition u of x, b equal to zero, so which is this first condition u of x, b equal to 0.

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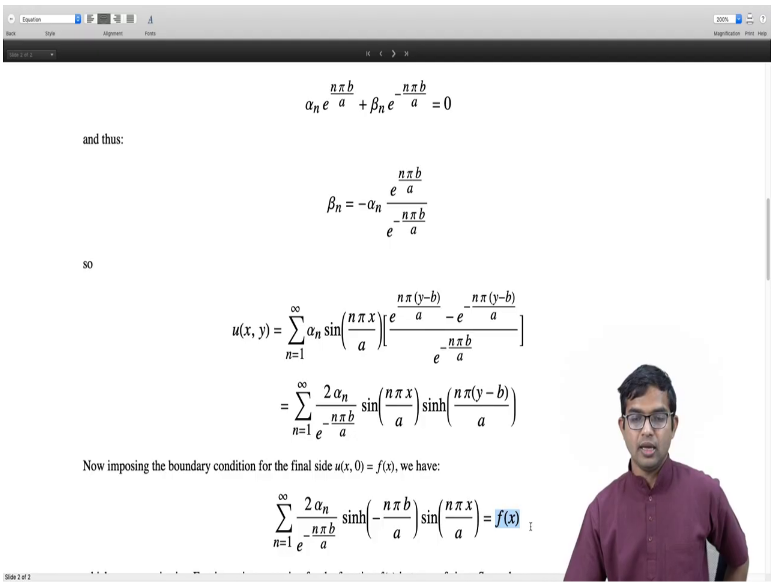

Then we see so, you know if you put y equal to b, so then we have alpha n times e to the n times pi b by a plus beta n times e to the minus n pi b by a, right. So, this has to be 0 for any value of x. So, it is not going to count at all. So, each of these factors must go to 0 and therefore, immediately tells us that beta n is related to alpha n in this manner.

So, a beta n is equal to minus alpha n times e to the n pi b by a the whole thing divided by e to the minus n pi b by a. So, u of x comma y is equal to summation over n going from 1 to infinity alpha n sin of n pi x by a times, you can actually pull out this alpha n because beta n is also really alpha n times this stuff.

So, then you can write this as you know e to the n pi y by a plus well it is a minus e to the minus n y n pi y by a the whole thing divided by e to the minus n pi b by a. Now that you know you can rewrite this you know bringing it to the numerator as well in this manner and then, you can pull out this alpha n. And therefore, you know if you multiply both numerator and denominator by 2 and so, you have 2 alpha n divided by e to the minus n pi b by a sin of n pi by pi x by a, you leave it as it is.

And then we have you know exponential of this stuff minus exponential of negative of the same stuff divided by 2. That is nothing but hyperbolic sin of n pi times y minus b the whole thing divided by a. So, now we have put in three of these boundary conditions. So, we just need to put in the final boundary conditions which is the whole function and that is why there is a whole set of coefficients n equal to 1 to infinity which are still unknown to us and all of that is contained in this function f of x which we can extract now.

So, the condition u of x, 0 equal to f of x is the same as saying you know f of x must be equal to this summation n equal to 1 to infinity two times alpha n divided by e to the minus n pi b by a times you know now this hyperbolic sin you know y is y is 0 here. So, we have n pi minus n pi b divided by a times sin of n pi x by a. So, this is actually nothing but this is actually nothing but a Fourier series representation of this function f of x, right.

So, f of x is defined in this interval from 0 to a, right. So, f of x is defined in the interval x, x will go from 0 to a, right. So, and therefore, it is being written in terms of sines right. So, if you are given just a function from 0 to a, it can be expressed in terms of sines or cosines. So, this is really a Fourier series expansion of a function which is defined in the interval from 0 to a. So, the way to think of this is to continue it to the interval from 0 to minus a.

If you take it to be an even function, then you will have to you will be able to express it in terms of cosines, but if you take it to be an odd function you will be able to express in terms of sines, but in the end if you are going to care only about the particular interval that you are interested in, it does not matter, right.

So, you can do either of these. Now f of x is here basically we already see that this is nothing but a representation of this function f of x in the interval 0 to a in terms of sines. So, this is nothing but a Fourier series expansion for which if we had been asked to work out this expansion for the function f of x, so the way to do this is you know to think of this entire thing as the coefficient corresponding to sine of that particular frequency.

So, we know that the way to solve this is to multiply throughout with just sin of n pi x by a and then integrate from 0 to a. You know all of these terms except one of them that we are interested in we will just vanish, right and then you will get the particular coefficient that we are after.

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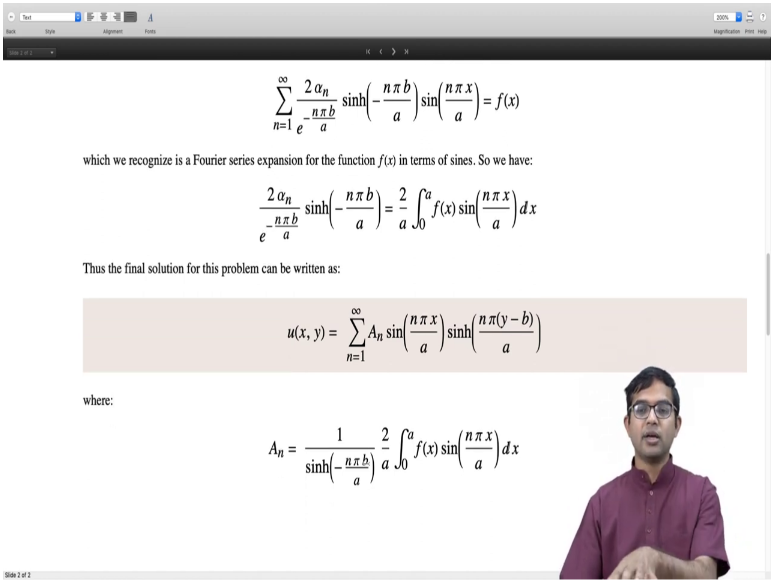

So, and that is that we can simply write this particular coefficient as 2 by a times integral from 0 to a f of x times sin of n pi x by a dx, right. So, this is nothing but an expansion. This is the trick we are using which you work out Fourier coefficients. So, now we are almost done. So, we just basically managed to work out the final solution.

Now u of x, y is nothing but summation over n going from 1 to infinity a n times sin of n pi x by a times hyperbolic sin of n pi times y minus b divided by a, where these coefficients a n are given by this whole stuff, right. So, this capital A n is actually nothing but the small two times alpha n.

So, yeah we have to be careful about this factor and so, in fact A n is given by just 1 divided by hyperbolic sin of minus n pi d by a times 2 by a times this integral 0 to a f of x sin of n pi x by a dx, right. So, you know this whole stuff. So, let us see, so we have this expansion.

I mean this is the expansion that we are after and then, we came to here and then we managed to equate these coefficients and then basically we managed to show that this the new coefficient, the final coefficients A n are simply written in terms of this expansion, right.

So, you should just go over this and convince yourself that indeed we have got all the factors right, but basically the idea is it is a combination of bringing in all the boundary conditions in a systematic way first of all choosing the right kind of solutions and then, also making sure that you know we extract the right kind of Fourier series. So, in this case, it is a sine Fourier series and then get the coefficients out. So, let us look at the other kind of boundary conditions where I have an example where you know the other kind of Fourier series also comes in.

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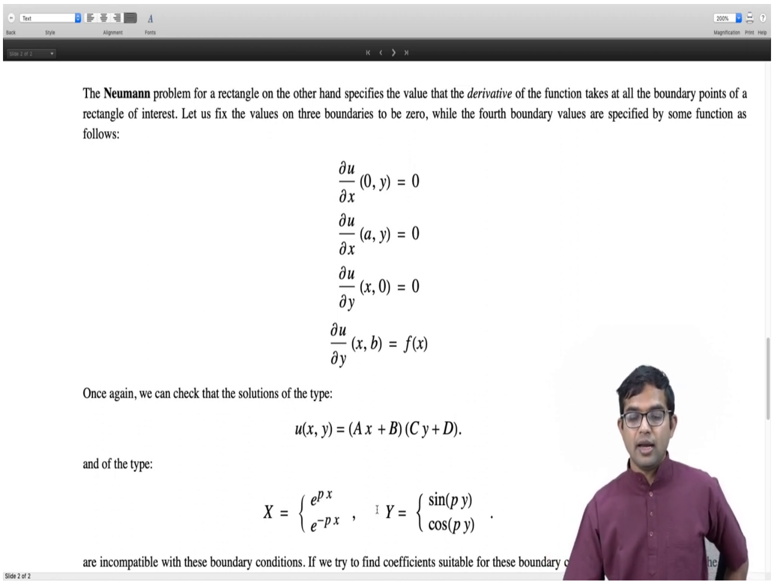

So, that is related to this Neumann problem. So, the Neumann problem for a rectangle specifies not the values of the function on the boundary, but the derivative of the function at all the boundary points or in some sense the gradient of this function, right. So, suppose we look at the same kind of a problem, but we specify dou u by dou x on for all values of f you know when x equal to 0 and for all values of y, right.

So, to be 0 that is you know dou u by dou x units are rectangular problem again and then, dou u by dou x again at the other end a comma y is also taken to be 0, but if you are going in the vertical direction, it is dou u by dou y that is specified. So, that is when x for all x, but y equal to 0 is also 0. So, along the x along the x axis, so we have this rectangle along the x axis dou u by dou y is taken to be 0. It is only in this stretch dou u by dou y along you know for all x, but at y equal to b is taken to be some function f of x.

So, this would be a Neumann problem. So, once again you know this is something that you can check explicitly: the other two cases of solutions do not contribute here, right. So, you can play with this, you can shift the boundary and then you will see that you know some other kind of solution will become relevant, right.

So, basically the idea is that you must see what opposite ends are. You know we desire for zeros to appear and you know there are two other ends one of which will be zero and the other one will be non-zero right. So, depending upon that you have to pick the right kind of solutions.

Again this linear solution times linear solution is not going to work out. You simply get the trivial solution a equal to 0, b equal to 0, c equal to 0 and d equal to 0 and you know if you choose exponentials along x and sines and cosines along y, you will see that for the Neumann problem, it is the derivative which needs to go at you know at these two ends. So, this is again going to fail.

And, but on the other hand if you choose the other condition which is if you choose sines and cosines along the x direction and you know exponentials along the y direction, you can set up the derivatives in such a way that you know one end you will go to 0 in the y direction that is at the bottom end and at the top end, it is going to be this function f of x, right.

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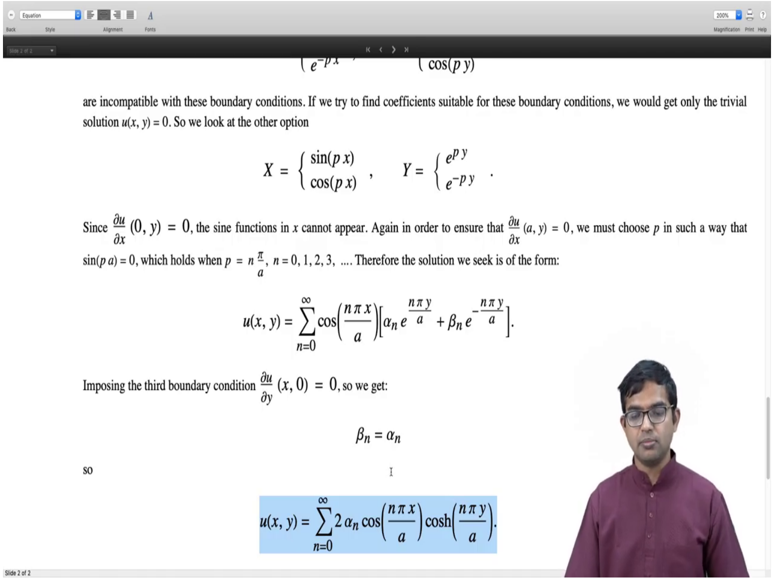

So, that is what we will look at now in some detail. So, you know if you have dou u by dou x for x equal to 0 and for all values of y to be 0, now the only way that can happen is if you know sin functions do not appear because it is about the derivative, right. If the sin functions appear, then if you take the derivative you will have cos functions and if you put x equal to 0, you will get a non-zero value, but here if you take the cosine functions, take the derivative and put x equal to 0 that is indeed going to go to 0.

And on the other hand if you want to ensure dou u by dou x of a, y to go to 0, then we must choose our p in such a way that sin of p a equal to 0, right. So, if you take cosine of p x take the derivative, then you will get sins of p s and then if you manage to choose your p in such a way that sin of p a equal to 0 which happens when p equal to n times pi by a n equal.

So, now 0 is also important because it is about the derivative. So, you start with the cosine function. So, that is going to give you a constant. So, as we will see, let me start with n equal to 0 1 2 3 and so on. Therefore, the solution we see is going to be of this form summation over n equal to 0 to infinity cosine of n pi x by a. So, if you put n equal to 0 here, you have a 1 right.

So, you have alpha n times e to the n pi y by a plus beta n times e to the minus n pi y by a, right. So, you can take this ansatz and quickly convince yourself that you know these two conditions hold this one and this one all automatically is taken care of if we choose this solution, right. So, two of these conditions are already done in order to do the other two conditions. There is one condition which is quite straightforward, right.

So, we have chosen you know the other end here to be to go to 0. So, it is actually an easier problem than the previous one in the sense that you know the relationship between alpha n and beta n turns out to be simpler here. So, if you choose dou u by dou y of x, 0 is equal to 0 you can quickly check that you know if you take the derivative of this function basically and dou u by dou y.

So, that is going to give you know this stuff and pi by a, then this stuff with a minus n pi by a and then, you have to put y equal to 0. So, immediately you will see that you will get beta n must be equal to alpha n. So, we have this simple form for our solution. It still needs to be completed because you have to put in the other boundary condition, but we are here. I have managed to multiply and divide by 2 and then rewrite this sum of these exponentials.

And then I have a divided by 2, I can rewrite it as hyperbolic cosine of n pi y by a. So, I have two times alpha n times cosine of n pi x by a times hyperbolic cosine of n pi y by a. Finally, we have this boundary condition for the fourth side which is dou u by dou y of x, b is equal to f of x.

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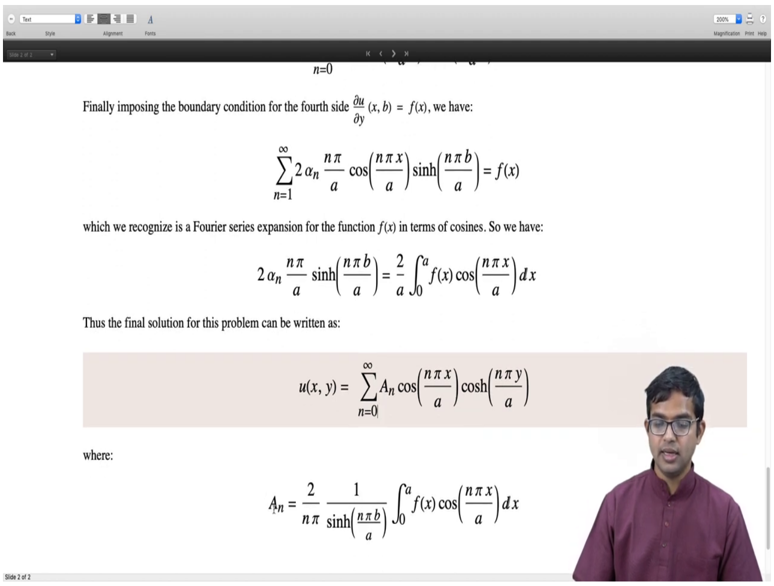

So, if I plug this condition in now I have this Fourier series representation for f of x, but in terms of cosines now, right. So, whether you get a cosine series representation or a sin series representation will depend on the boundary conditions involved. So, basically the idea is that you know your function f of x is defined in the interval 0 to a.

So, there is a way to describe this function in terms of you know cosines or sines because you can extend it to be an even function or you can extend it to be an odd function and then extract this Fourier series. And it does not matter you know as far as you are looking at only the interval zero to a is concerned, both are completely legitimate solutions.

So, in this case it is a Fourier series in cosine functions and so, therefore I mean we have this n pi by a also which has come in because we have to take this derivative, right. So, when you do dou u by dou y, so you know this n pi by a comes out and then, cos becomes hyperbolic sign you have to check all of this. It is just some elementary calculation involved and then finally, we simply recognize all this stuff times, this stuff as really like the coefficients of this cosine series.

So, all this stuff which is really the coefficient is nothing, but 2 by a times integral 0 to a f of x times cosine of n pi x divided by a dx, right. So, this is the theory of Fourier series representation, right. So, you can multiply throughout with this function and convince yourself that indeed if you have cosine square, that is going to give you a non-zero value and specifically it is going to give you this normalization.

But, you know these are in some sense orthogonal functions, right. So, if you multiply you know cos two cosines with different frequencies and then integrate from 0 to a, those are all such terms will vanish and so, finally the problem can be written. The final solutions can be written as u of x, y summation actually this n needs to go from 0 to infinity.

So, it will go from 0 to infinity a n cosine of n pi x by a hyperbolic cosine of n pi y by a and you can convince yourself that indeed a n is nothing but 2 divided by n pi times 1 over hyperbolic sine of n pi b by a times integral 0 to a f of x times cosine of n pi x by a dx, ok. That is all for this lecture.

Thank you.