Mathematical Methods 2

Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Module - 07
Partial Differential Equations
Lecture - 65
The Laplace Equation

Ok, so we have been studying Partial Differential Equations. We looked at some very general aspects and how to classify partial differential equations using the discriminant of the quadratic form involved. So, in this lecture we will look at a specific PDE which is of great importance in Physics. It finds applications in a number of fields in Physics and Engineering. And so, we will start with some very elementary properties of this differential equation and then, we will see how to solve this equation in the following lectures, ok.

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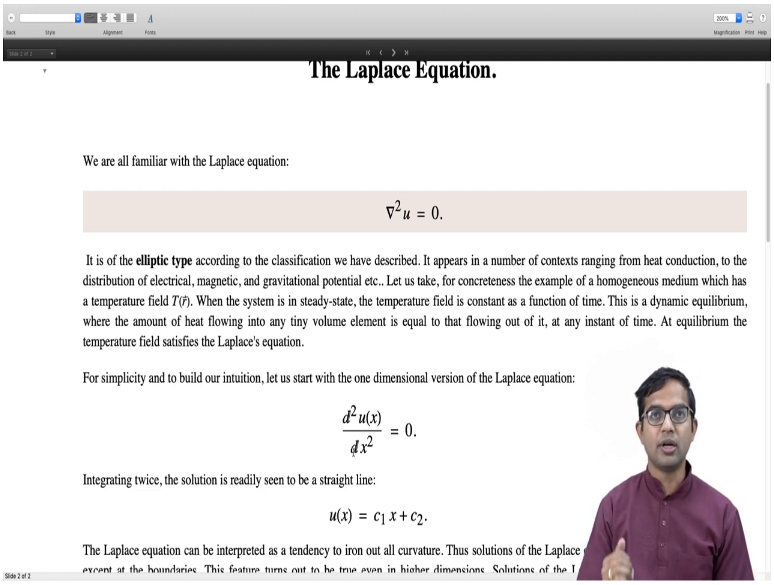

So, the Laplace equation has this you know Laplace you know operator del squared and it acts on some function which can be a function of one variable, two variables, three variables or even more sometimes. But usually we restrict our attention to two variables or three variables in you know x, y and z is as far as we go in Physics, right. It is of the elliptic type right according to the classification we have seen.

So, elliptic type comes about when you know dou squared u by dou x squared and dou squared u by dou y squared, both of them have the same coefficients, same sign, same magnitude same which is just one in this case. And so, it appears in a number of contexts. So, like we said, heat conduction is one of them. Then also you might have encountered this in the context of you know electrostatics, electrodynamics magnetic potentials can be described using this differential equation.

So, yeah one concrete example is that of temperature fields, right. So, if you have a heat rod which has reached steady state. And so, often we are interested in finding out the temperature profile of a heat rod or some object with some specific geometric shape you know that is where many of the details of solving this kind of a PDE will come in depending upon the geometry and the boundary condition and also those things will come a bit later.

So, the point is that so it is a kind of a dynamic equilibrium is set up where you know the amount of heat flowing in into some region is equal to the amount of heat flowing out of it. Therefore, you know the temperature at any point in space is going to be a constant as a function of time and so, that is going to be described by this Laplace equation, right. So, if you are in 2 d or if you are in 3 d, it is a little more complicated.

So, what we will do is, we will start with the one dimensional version of the Laplace equation which is in fact a kind of a trivial differential equation. It is not even a partial differential equation; it is just a function of a single variable. So, we might as well write it as d squared by d x squared acting on u of x equal to 0. So, if you integrate it once you get d u by d x is equal to a constant and then, if you integrate it a second time, you will get c 1 x plus c 2 where c 1 and c 2 are constants which are available for you to fix based on you know boundary conditions now.

So, yeah so the solution is a straight line and so, already there are there is you know there is insight which we can pull out from this the 1 d version and which we will see we will carry to higher dimension as well.

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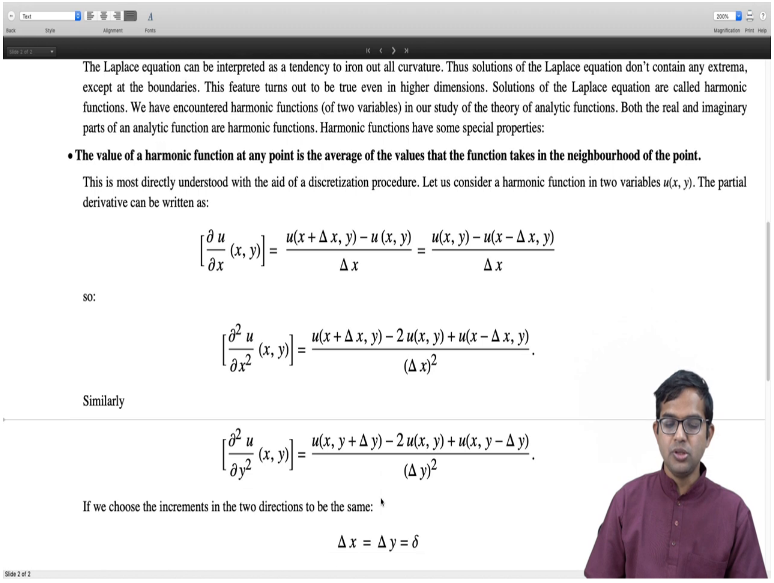

So, the key point is that this equation is explicitly a tendency to iron out all curvature, right. So, the second derivative is where the information about the curvature of the function is contained. So, we know that d u by d x is the slope d squared d u by d x squared is the curvature. So, clearly the solution does not contain any quadratic part, so it is a straight line, right.

So, there is another way to think of this feature. I mean straight line means that basically there are no maxima or minima sitting in you know in the bulk, right. So, of course there is at the edges of course right at the top of the line and at the bottom of the line. It is going to be you know the maximum and a minimum right are possible at the edges, but within the you know center, there is no curvature right.

So, the line is not allowed to bend right which is in some sense explicitly forbidden by this condition and therefore, it is a you know it is going to give you no maxima or minima, right. So, another way of thinking about this is that in fact you know the value of this function at any point is going to be the mean of you know the value of the function in its neighborhood, right.

So, if you are sitting in a straight line, clearly the value of the function at any point is going to be the average of the value of the function. It is slightly above and slightly below right. So, I mean if you take the same distance above and the same distance below, right. So, there is a you know more rigorous way of stating this and so, we will see this as we go along I mean although I mean we will state it in a more general fashion perhaps not with complete rigor, but it is possible to state this property.

And it is in some sense these two properties are closely related. One is the fact that there is no curvature which means that you cannot have maxima or minima and so that also is in some sense equivalent to the fact that you know you have this mean value property, right. So, this it turns out actually carries forward to higher dimensions as well. So, in two dimensions we will revisit this same couple of properties a little bit ahead in this lecture.

But I mean it is also useful to point out that in fact we have already encountered functions which satisfy the Laplace equation functions of two variables which satisfy the Laplace equation we have already encountered, right. So, these are the real and the complex parts of analytic functions, right. So, we have seen that analytic functions are made up of two functions of two real variables.

So, there is a real part which we have often called u of x comma y and then there is the imaginary part which we called v of x comma y and each of these you know the Cauchy Riemann conditions must hold if these the overall function is analytic. And therefore, individually you can show that dou squared u by dou x squared plus dou squared u by dou y squared equal to 0 and some similar equation also holds for the imaginary part.

So, functions which satisfy the Laplace equation are called Harmonic functions. We have seen that analytic functions the real and imaginary parts, both are separately harmonic functions. So, now harmonic functions like we just said earlier with the 1 d case satisfy these two crucial properties. One of them is the value of the harmonic function at any point is the average or the values that the function takes in the neighborhood of that point, right.

So, let us understand this for the 2 d version using a discretization of the differential equations. So, you have this partial differential equation. So, let us take this partial differential equation and then rewrite you know or recast it into a discrete version of the same PDE where it becomes explicit how this average value property comes about, right. So, you know the partial derivative is really you know u of x plus delta x comma y minus u of x comma y the whole thing divided by delta x in the limit of delta x becoming arbitrarily small, right.

So, if you are doing a numerical way of computing this, then you will make delta x small, but it is not going to go to 0 right. You can make it very small, but in practice it is going to be finite and so, you can either take the forward difference or the backward difference. Both of these are completely legitimate ways of working out the partial derivative of this function at the point x comma y and in fact, both of them have to converge through the same point right.

So, that is how you have a meaningful definition of the partial derivative. Now if you go ahead and do a second derivative, right. So, it is convenient to use you know the backward difference in the second one, if you have used the forward difference in the first one or you know backwards difference followed by forward difference, so that you know we get this nice form. So, for example if I look at the first expression here and then use the backwards difference to compute the second order derivative, then I will take this whole thing as it is.

I will just rewrite it here minus u of x comma y minus u of x minus delta x comma y the whole thing divided by you know delta x, then the whole thing itself has to be divided by delta x. So, that gives me a delta x squared in the denominator and then there are these 2 u of x comma ys which come in. So, I have a u of x plus delta x comma y minus two times u of x comma y plus u of x comma x minus delta x comma y.

So, this is a you know this is the discrete version of the partial second derivative with respect to x. We can also write a similar expression for the second order partial derivative of u with respect to y. So, then we will have you know u of x comma y plus delta y minus two times u of x comma y plus u of x comma y minus delta y the whole thing divided by delta y the whole squared.

So, of course we intend to take the limit of delta x and delta y becoming arbitrarily small. Suppose we choose these two increments to be the same delta, let me call this as delta.

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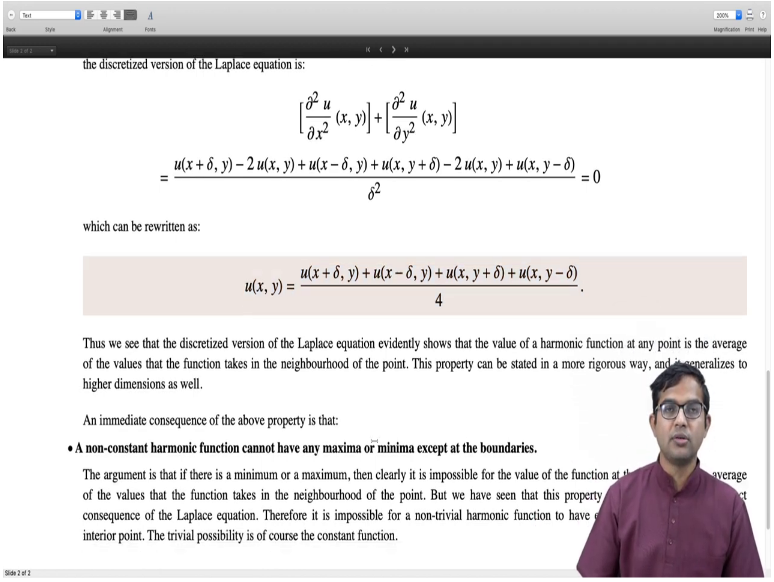

And then rewrite this the Laplace equation in terms of you know it is the sum of these two quantities. So, if I write this discrete version, I have this longer expression. So, u of x plus delta comma y minus two times u of x comma y plus u of x minus delta comma y plus the whole thing again similarly, u of x comma y plus delta minus two times u of x comma y plus u of x comma y minus delta. The whole thing has to be divided by delta squared and if this has to be 0 because that is the Laplace equation.

Now you can rearrange things and rewrite this as u of x comma y right. So, you can pull out. So, it is a minus 2 u of x comma y and a minus 2 x comma y. You bring both of these to the left hand side and then divide by 4. Then you see that you get u of x plus delta comma y plus u of x minus delta comma y plus u of x comma y plus delta plus u of x comma y minus delta the whole thing divided by 4.

Basically what we are doing is, we are looking at this function u of x comma y on a grid right. Although I mean the continuous version is where we started. So, suppose we make a very fine grid of points. So, what you have managed to show is the condition that del squared u equal to 0. So, the Laplace equation condition is identical to saying that the value of this function u of x comma y at any grid point is equal to the average of the values which the function takes at all the neighboring points, right.

So, this property is in fact true even in the continuous version itself. You can come up with some way of writing an integral and show that the value of u of x comma y is going to be the average of the value that it takes in some neighborhood. You can come up with some path and then work it out and so on. There is a more rigorous way of showing this, but here we see explicitly and in a simple way that indeed there is this property of the mean value you know of the function taking the mean value of its values in its neighborhood.

And this is in fact identical to the property of you know of the definition of the Laplace question, right. So, this is completely equivalent to saying that you are solving for del squared u is equal to 0 and so, this is in fact a way to solve the Laplace equation numerically, right. So, in fact you can see exact solutions of the Laplace equation in 2 d and maybe 3 d as well.

Can one extrapolate some of these methods? So, 2 d is actually a great place to do, I mean you can try and work this same kind of a property out for 3 d and convince yourself that indeed you will have some kind of a cube and you will get the average of you know u of x comma y will turn out to be the average of all the nearest neighboring points in the in three dimensional space as well, right

But so, the key point I want to make here is that this is a way to numerically solve this PDE. So, suppose you are given some complicated geometrical surface right and if you are given boundary conditions, you just make a grid and then put the value of the function u of x comma y to be 0 at all points and then change the boundary conditions alone to the values that are given there. So, if you have set the boundary conditions and then you just simply go to every point in your grid, make the grid as fine as you can as fine as your numerical, you know resources will allow you.

And then simply keep on replacing every grid point's value by the average of its nearest neighbors. So, initially of course there will be lots of zeros. So, many of these updates will probably return 0, but slowly you will start seeing that the values from the boundary will start showing up in all the points of the grid. And this is going to be a point where you know when convergence is attained, you can actually match the analytical solution with the numerical solution and you will see that they will be basically this same, right.

So, this is in fact it is a legitimate numerical technique to solve the Laplace equation. So, yeah you can think of you know a consequence of this is the next property or you can also think of it in the other direction. Both are really equivalent to each other in some sense. So, the fact that every point's value is equal to the average of its nearest neighbors immediately implies that there is no chance that this function can have any maximum or minimum in the bulk, right.

Because if it does have a maximum, that means that its value if it is a maximum for example, then the value of this function at that point is strictly greater than the value of the function in all its neighboring points. Therefore, the average of all of these is also going to be less than the value of this function at that point. So, it cannot satisfy this property.

So, the fact that you know there is this mean value property is completely equivalent to the fact that a harmonic function cannot have any maxima or minima except the boundaries unless you have the trivial solution. If you have a trivial solution where the harmonic function is just simply a constant, that is the only case where you can have you know then every point is a maximum or every point is a minimum or you know it's a very special and trivial type of scenario.

But in general these two properties do hold for harmonic functions and they are completely equivalent. That is all for this lecture.

Thank you.