Mathematical Methods 2

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Module - 07
Partial Differential Equations
Lecture - 63
Canonical Forms for Parabolic PDEs

So we continue our discussion of Partial Differential Equations. We have been looking at this classification of second order PDEs, and how it is possible to make a suitable transformation to put the PDE into a canonical form. So, in this lecture, we will concentrate on parabolic PDEs and work out the method by which we can write it in the Canonical Form.

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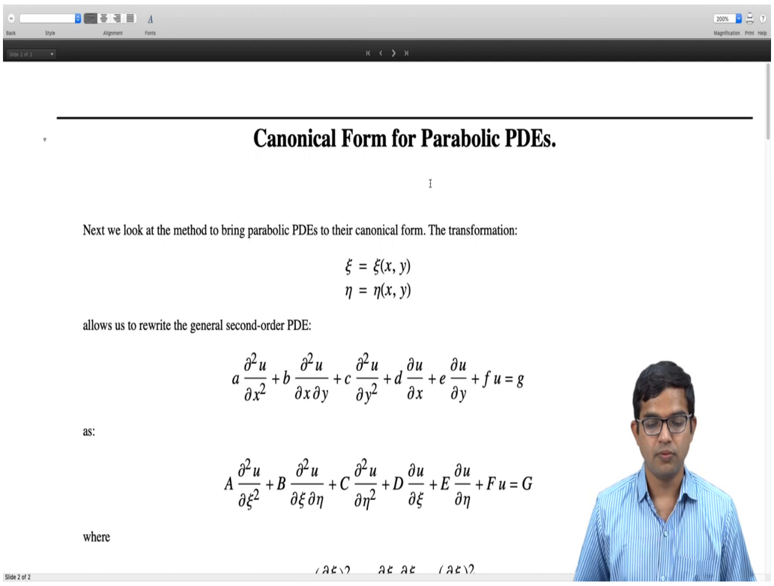

Ok. So, the main method is to come up with a suitable transformation zeta of x comma y, and eta of x comma y. And in terms of these new coordinates, we will be able to rewrite this general second order PDE in terms of another second order PDE which basically looks you know in form almost like the starting point except that we have the power to choose these coefficients in a manner which is most convenient for the type of differential equation that we are studying.

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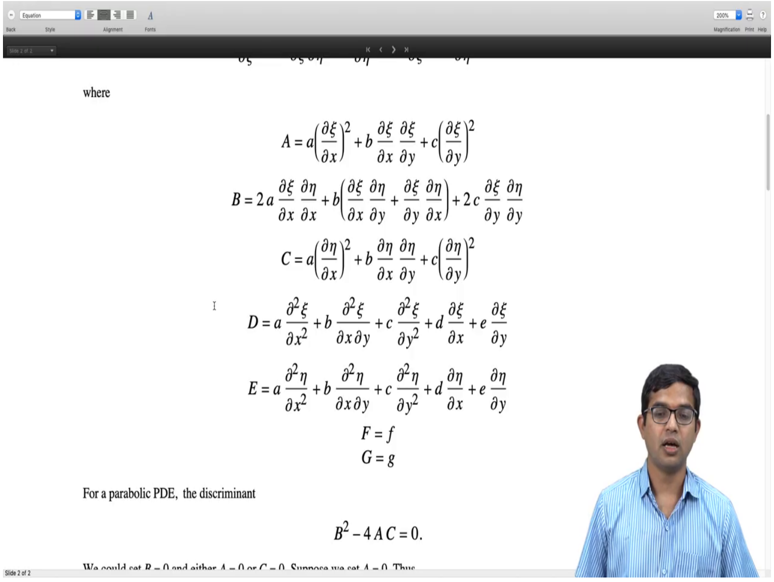

And so and then we have worked out this general prescription by which capital A is related to all these small a, b, c, d, e, f, g. And these partial in terms of these partial derivatives like here B is given in the in terms of this relation; C is given in terms of this relation; D is given in terms of this relation; E, F and G are all written down right, so which is something which we saw a couple of lectures ago.

Once again if you have not convinced yourself of this, now is a good opportunity to go back and check this and see for yourself that indeed this is the correct way to go from one basis to another basis right. So, it is like one set of coordinates to another set of coordinates.

So, in this lecture we are looking at parabolic PDEs right. So, attention is focused on parabolic PDEs for which the discriminant is 0 does not matter which whether you are you know looking at the discriminant in terms of this small b and c, or in terms of this capital A, B and C, the discriminant is simply given by capital B squared minus 4 AC or small b squared minus 4 small a c both of which are going to be 0.

Now, so like we said earlier the type of the differential equation is just decided by a, b, and c. So, the other coefficients do not really matter as far as the type of the differential equation is concerned.

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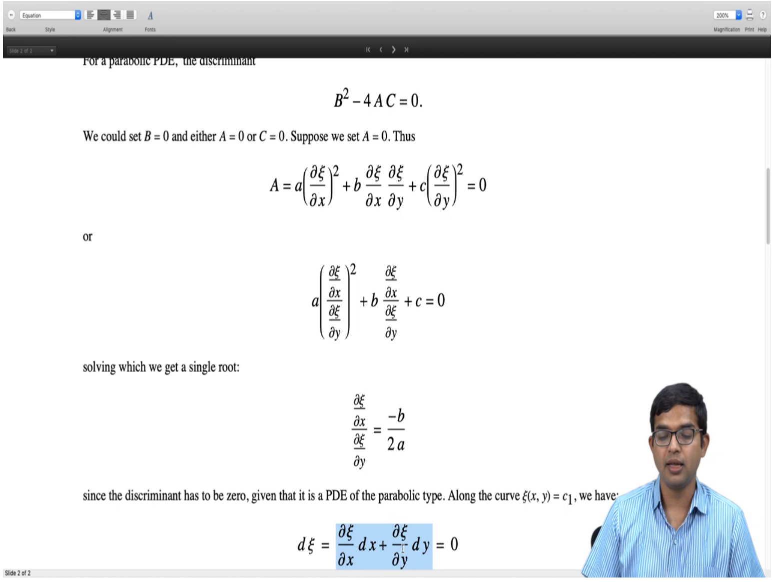

And, so in this case, it is convenient to actually to set B equal to 0, and one of these two either A or C to be 0 right. So, if we do this, then automatically it means that the discriminant is 0. I mean when we start with PDE of the parabolic type, of course, we may not have this form where either A or C is 0. In fact, all three may be present small a, small b and small c right, in general, it is true that all three are present.

So, the idea is that if you can take it to a new PDE in which small b capital B is 0, and either A or C, you know or 0 then you are going to get a PDE which has a much simpler form which will be in the canonical form as we will see. Now, suppose we set capital A equal to 0.

So, what it means is we said this quantity you know according to this transformation, we will set this to be equal to 0, a times dou zeta by dou x whole square plus beta dou zeta by dou x times dou zeta by dou y plus c times dou zeta by dou y the whole squared equal to 0 right.

So, this corresponds to this quadratic equation in this ratio. So, I can divide throughout by dou zeta by dou y the squared. So, I get this quadratic equation in this ratio dou zeta by dou x the whole thing divided by dou zeta by dou y. And which can be solved right, we know how to solve a quadratic equation.

So, if we solve this quadratic equation, we are going to get only one root right because this is of parabolic type right. So, we know that the discriminant is 0 for this type of a PDE. So, therefore, you know minus B plus or minus square root of discriminate, there is no plus or minus: if you add 0 plus 0 or minus 0, it is the same. So, there is only one root you get for this, there are not two roots like we had in the hyperbolic case.

So, once you get this single root, we can actually go ahead and you know connected to this curve zeta of x comma y equal to c 1 right. In a manner very similar to how I did with hyperbolic type. So, we consider this curve zeta of x comma y equal to some constant c 1 we call it.

Suppose, we take you know differential of this d zeta is you going to be 0 because along this curve you know the values are constant, but we can write this d zeta is dou zeta by dou x times dx plus dou zeta by dou y times dy which in turn allows us to work out the slope along this curve d b by dy by dx is given by minus of this ratio, but this ratio is something we already know right. So, this ratio is minus b by 2 a. So, we get a plus b by 2 a for dy by dx.

So, one of the two new coordinates can be found by simply solving for this differential equation dy by dx is equal to b by 2 a, we solve for this. And then we will get a curve which corresponds to zeta of x comma y is equal to c 1. And then from which we can read off this new coordinate zeta of x comma y.

And so for the parabolic PDE using this technique, we are going to get one, one coordinate right. So, we still have to get a second coordinate, but we will see in a moment how that is actually fairly straightforward to get the other coordinate.

But before we do that let us quickly show that in fact capital A equal to 0 automatically implies that B equal to 0. We do not need to put this capital B equal to 0 separately in this condition. So, if you can choose capital A equal to 0 in an equation of parabolic type, then automatically capital B is also going to be 0. So, the way to argue is as follows. So, we know that capital B, according to this you know box of relations that we have worked out which is completely general, is given by this expression.

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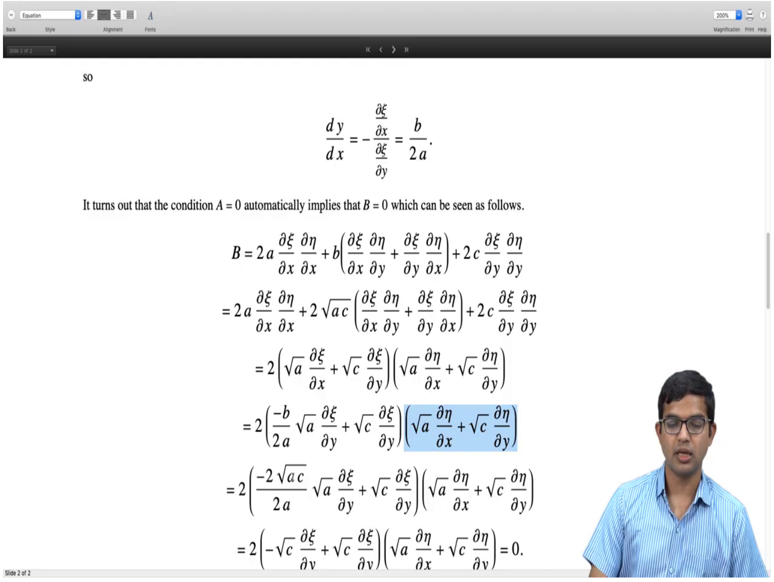

So, we have capital B is equal to 2 a times dou zeta by dou x times dou eta by dou x plus b times dou zeta by dou x times dou eta by dou y plus dou zeta by dou y times dou zeta by dou x the whole thing plus 2 time c times dou zeta by dou y times dou zeta by dou y right. So, this is the general expression for B.

Now, we can rewrite this as I mean the first term can remain as it is, but in place of b, we can write two times square root of ac right, so because this is a differential PDE of a parabolic type. So, b squared is equal to 4 small ac which is. So, therefore, b is equal to 2 time square root of ac, and then all this stuff inside the bracket remains as it is then we have plus 2 c.

So, now, it is convenient to pull out this factor of 2. And then you know rewrite this as a square root of a time dou zeta by dou x plus square root of c times dou zeta by dou y time square root of a time dou eta by dou x plus square root of c times dou eta by dou y. You can quickly check that indeed we have just factorized this expression right. So, the way to check this is to actually multiply these two and then you will see that you get back this whole stuff.

Now, because of the fact that this is a parabolic PDE, we can rewrite this, well, I mean and also the fact that dou zeta we already know the ratio of these dou zeta by dou x divided by dou zeta by dou y. We can rewrite this dou zeta by dou x in terms of dou zeta by dou y times minus v by 2 a right, so that comes from here. So, dou zeta by dou x that is actually anything but minus b by 2 a times dou zeta by dou y.

So, if I replace this dou zeta by dou x by dou zeta by dou y times this factor minus b by 2, then I can actually simplify this part. And then I see that once again it is convenient to rewrite this b as 2 times square root of ac, then we have many simplifications.

So, this square root of a divide and below will have some cancellation. So, a and will be just left with minus square root of c times dou zeta by dou y plus square root of c times dou zeta by dou y. So, actually these two terms will cancel and give you a 0. And so effectively or multiplying by 0, therefore, all gives you just a 0.

So, what we have managed to show is if you choose capital A to be 0, then automatically it implies that capital B is also 0. So, we do not have to separately choose capital B, B to 0, it automatically comes from the parabolic nature of this PDE right. So, thus from all this you know from putting capital A equal to 0, we have managed to just get one differential equation, one condition so or one characteristic or zeta of x comma y equal to c 1. So, we can get zeta of x comma y.

So, how do we get the other coordinate? So, it turns out that you can actually choose any other coordinate you want as long as it is not parallel to the first coordinate. What does that mean?

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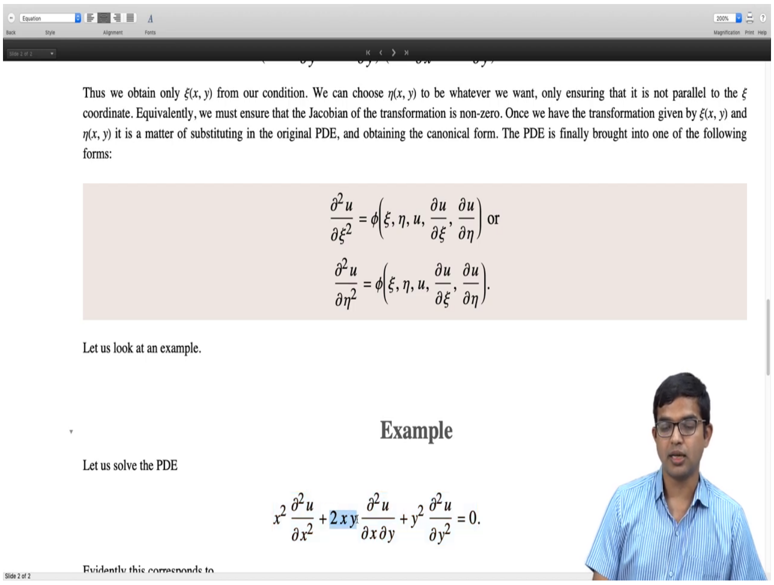

It means that we must choose another coordinate such that the Jacobian of these two is nonzero right. So, we have to make a transformation and you just have to ensure that the

Jacobian is nonzero, so that you know whatever information is contained in terms of these two variables x comma y will be written in terms of a new two variables.

You cannot compress information of content of two independent variables in terms of just one independent variable, you still need two independent variables. So, zeta and eta must be independent. So, we can choose another variable arbitrarily, and it will all go through right. So, we will be able to rewrite it in terms of one of these two canonical forms.

So, finally, we will be able to just write it as dou squared u by dou zeta squared is some function of all the stuff. It can be quite complicated, but you will be able to write it either in this form or in this other form: dou squared u by dou zeta square dou eta square is equal to this complicated function, potentially complicated function or a simple function. So, all of this again is best illustrated with the aid of an example.

So, let us look at an example: suppose we start with this p d v, PDE x square times dou squared u by dou x squared plus 2 x y time dou square u by dou x dou y plus y squared times dou dou squared u by dou y squared equal to 0.

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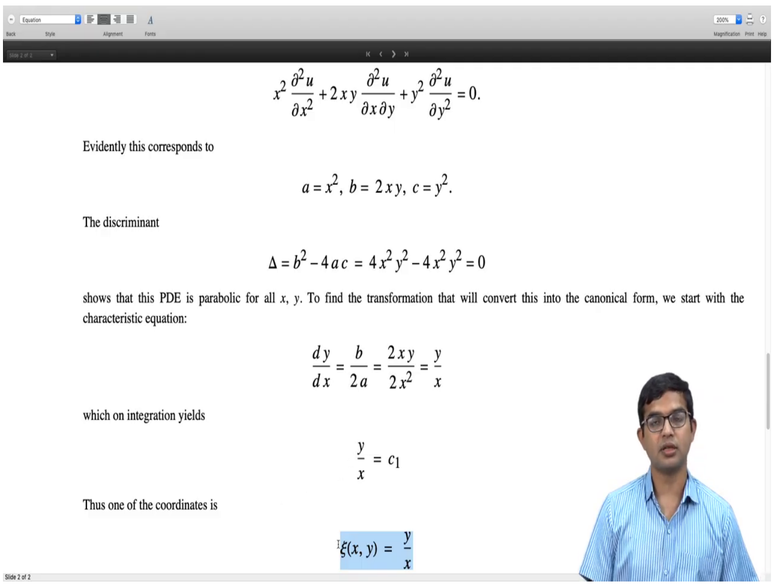

So, this corresponds to a being equal to x squared, b being equal to 2 x y, and c being equal to y squared, and then of course, d, e, f, g, all of them are 0. We have taken a relatively simple PDE to start with. And so the discriminant we see is just b square minus four ac which is 4 four x squared y squared minus 4 x squared y squared which is 0 for all values of x and y. So, this is a PDE of parabolic type for all x comma y.

And to find the transformation that will convert this into the canonical form, we start with the characteristic equation. There is only one characteristic equation: it is of parabolic type. So, dy by dx should be equal to b by 2 a. So, in this case b is 2 x y, and then 2 a is 2 x squared so which is equal y by x which is a straightforward differential equation to solve, dy by dx equal to y by x we get a straight line y is equal to c 1 times x or y by x is equal to c 1.

And so the one of the coordinates is actually zeta of x comma y is equal to y by x right. So, this is a family of curves which pass through x comma y. And so then we just read off from here, here that the transformation of one of the coordinates to be chosen is zeta of x comma y equal to y by x.

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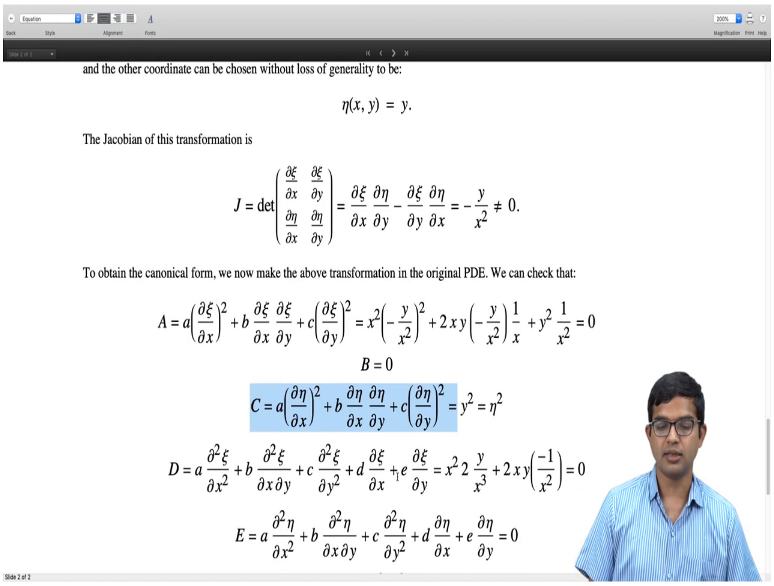

So, now we can have the freedom to choose the other coordinate you know in lots of different ways. So, one simple way to choose this eta of x comma y is to just choose it to be y. So, if we do this, then we can quickly check that the Jacobian is given by this determinant.

So, dou zeta by dou x is so times dou eta by dou y, so it is my dou zeta by dou x will give you minus y by squared times 1 so minus y by squared, but the other one is dou eta by dou x is 0. So, this part will not even contribute. So, we are just left with minus y by x squared which is not 0 in general.

Therefore, to obtain the canonical form, we have to make this transformation right. So, you can check that capital A will be 0, we have worked this out, we have said that on general principles this has to be. So, you can check that indeed this is true for this particular choice of zeta and eta explicitly you can check this is 0, and B is 0, and C here it turns out to be y square which can be written as eta squared.

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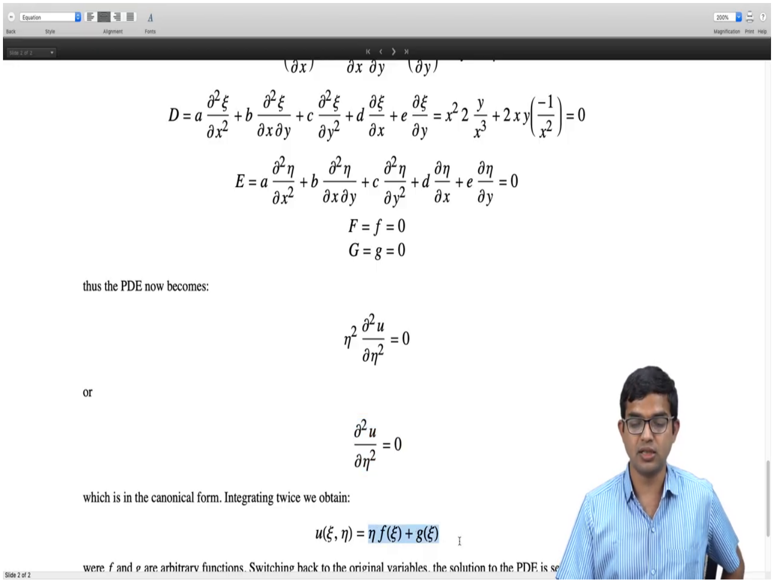

And then D, E, F, G all of them will be 0 as you can check right. So, this requires a little bit of algebra, but you can convince yourself that indeed this is true, this is 0. And this stuff is anyway 0, F, G are also 0. So, the PDE becomes actually very simple in the scale is just eta squared times dou squared u by dou eta squared is 0 or equivalently is just dou squared u by dou eta squared equal to 0. It is as simple as this which is indeed in one of the canonical forms for parabolic PDE.

And then if you have to you can actually write down the solution in this case in very general terms. So, if you integrate it twice, if you integrate it once, you will get a you know function f of zeta right, you can be an arbitrary function of zeta is basically a constant. And then if you integrate a second time, you will get a eta times f of zeta plus another constant as far as etas concern which can be a function of zeta. So, this is the general solution.

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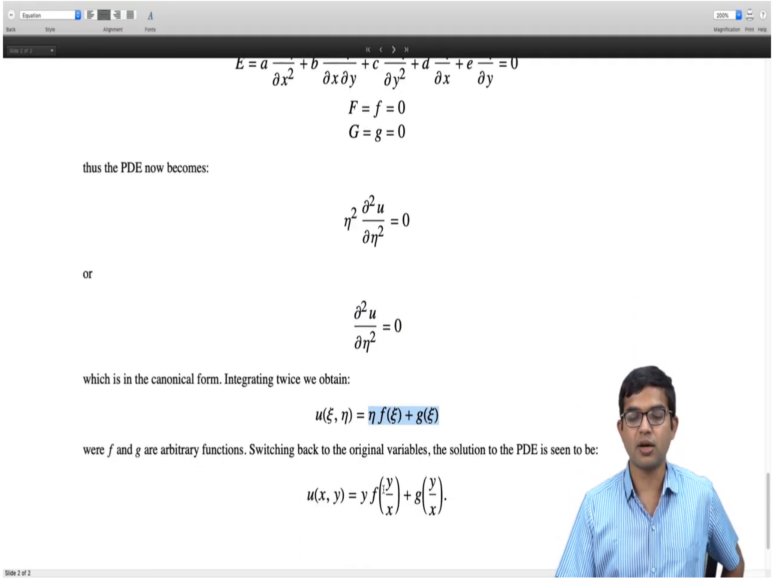

And if we switch back to the original variables, you can write down the solution as u of x comma y, y is equal to y times f of y by x plus g of y by x ok. So, that is all for this lecture.

Thank you.