Mathematical Methods 2

Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Module - 07
Partial Differential Equations
Lecture - 61
Classification of Second Order PDEs

So with this lecture we begin the next topic namely Partial Differential Equations. So, the number of sources we will dip into and you know extract inspiration from. So, one of the books which I really like is a sort of an exhaustive treatment by K Shankara Rao.

So, its called I think Introduction to partial differential equations there is also one chapter in Ghatak Chuva and another co-worker's textbook on Mathematical Physics then there is also Joglekar's book we might dip into for some parts then there is Balakrishnan's book - this also has some useful perspectives to offer, then of course, we have books by Boas and Arfken right.

So, we will dip into many sources, but in this lecture we will start with a classification of you know we are primarily going to be interested in second order PDEs because that is where you know the attention in physics is largely focused on and so, of course, you know problems in their you know most natural form when they rise in a physics application are generally partial differential equation. We studied ordinary differential equations you know in great detail in the previous version of this course.

And so, but really you know in problems have their origin in which have their origin in physics applications are usually partial derivative partial differential equation in nature and then it certainly from a theoretical point of view, it helps to have solid understanding of ODEs before we embark on our study of PDEs. So, in this lecture we will sort of set up the scene for second order PDEs and how there is this very useful classification for them ok.

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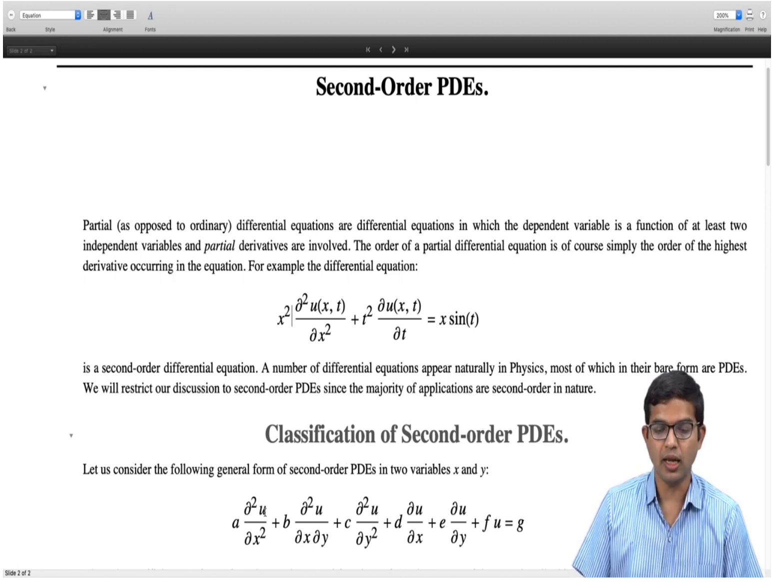

So, partial differential equations as I am sure you know we all know are differential equations in which there are partial derivatives involved right. So, there are at least two independent variables and partial derivatives are involved. So, the order of a partial differential equation is of course, somewhat similar to the idea of the order of an ordinary differential equation and it is given by the order of the highest derivative occurring in the equation.

So, I have cooked up this partial differential equation. So, x squared times the second order partial derivative with respect to x of this dependent variable u of x comma t. So, u is a function of both x and t and then plus t square times dou u by dou t. So, it is the first order derivative which appears here is equal to some function involving both x and t right.

So, we see that indeed the highest order derivative which appears here is with respect to x and its of order two therefore, this is a second order partial differential equation.

So, we will concentrate on second order PDE’s and in particular we will specialize into certain you know very well studied kinds of partial differential equations, but before we do that we have this sort of broad classification of second order PDEs and we will talk about what is called a canonical form and so, we will you know break these into three different classes and so, special treatment. So, each of these kinds will then follow right.

So, in general partial differential equation solving PDEs is a hard problem. So, particularly if you have to put in boundary conditions that is a whole machinery associated with the solution of PDEs right. So, we will start our discussion with some sort of broad ideas and so, there this transformation possible. So, we will discuss these transformations and come up with what is called a characteristic equation right.

So, which is all within this umbrella of classification of these second order PDE's. Once we have done this sort of a broad level discussion we will also go into the machinery of how to solve some of some PDEs which appear in specific physics problems.

So, that is coming up later. So, let us consider this sort of general second order PDE involving two variables x and y right. So, u is a function of both x and y, I have not explicitly written u of x comma y in order to keep you know not clatter this equation too much. So, I have a times dou squared u by to x squared plus b times dou squared u by dou x dou y plus c dou squared u by dou y squared plus d dou u by dou x plus e dou u by dou y plus f u equal to g.

So, this is a very general second order PDE involving two variables right. So, in real world applications sometimes it's useful to consider three variables or even four right. So, four variables is the general problem involving four variables already becomes quite complicated, but up to 2 and 3. So, there is this sort of general framework which is possible to write down and there is this canonical form and so on which can be done.

So, but from our point of view we will just stick to the two variable version. So, second order PDEs and where the dependent variable is a function of two independent variables I am calling them x and y. So, these coefficients a, b, c, d, e, f and g are all in general functions of both x and y right.

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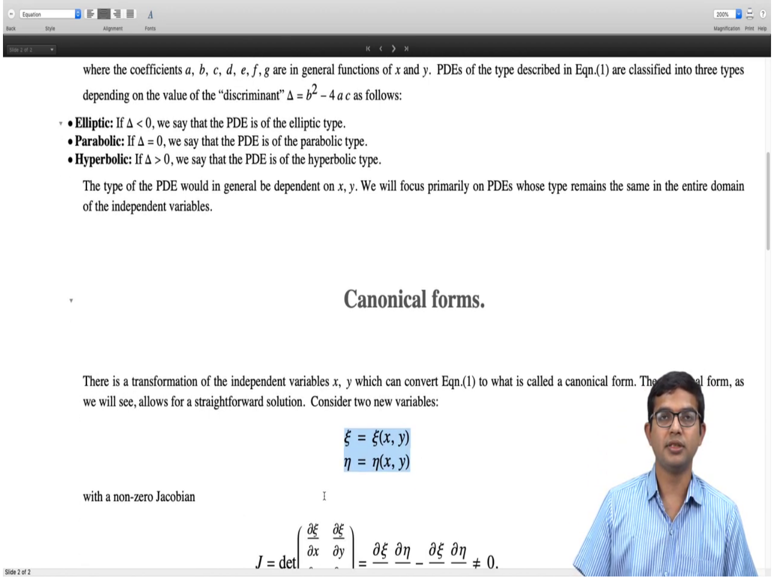

So, now, if we consider a generic PDE of this form, it's possible to classify into three types based on the relationship between a b and c right. It turns out that these other terms you know do not really feature in this classification right; it's only about a, b and c. So, you sort of think of this as a quadratic form and then you study the nature of the so- called discriminant right.

So, you will see in a moment how this appears right when we further look at some transformations. So, the idea behind classifying these differential equations into these three different boxes has to do with a useful transformation which can bring these differential equations into a canonical form.

And this canonical form will actually depend only on these three terms; the others also will undergo some changes, but really the focus is on a, b and c. So, you define this discriminant delta which is given by b squared minus 4 ac - you can think of it like a sort of a quadratic equation will also come in explicitly, but this is like a quadratic form.

So, the discriminant delta is made up of d squared and 4 ac d squared minus 4 a c. Depending upon the value of this discriminant delta, there are these three different types possible. So, if delta is less than 0 if b squared minus 4 ac turns out to be less than 0, then we say that the PDE is of the elliptic type.

If delta equal to 0 that is if b squared happens to be equal to 4 a c, then we say that this is of the parabolic type and if delta is greater than 0 we say that the a partial differential equation if the is of the hyperbolic type. So b squared is greater than 4 ac. Now of course, I said a, b, c can all in general be functions of x and y. So, indeed. So, the type of the differential equation itself can vary as a function of x and y right.

So, we will you know primary look at PDE's whose type remains the same in the entire domain in which these independent variables are defined, but in general it is of course, true that you know a differential equation could be elliptic in one region, but hyperbolic in some other region and parabolic in a third region and so on right.

So, this quantity delta is really computed at the point x comma y right. So, we will see that there are examples of you know generic classes where the value of delta is the same for all values of x comma y or its sign is the same right its all about the sign whether its positive, negative or 0 ok.

So, let us look at the idea of a canonical form. So, there is. So, the idea is that once you have figured out the type based on you know a, b and c you can transform this differential equation this partial differential equation to into another partial differential equation, you know where you basically make a change of variables right.

So, what we do is, you know, consider two new variables and recast this PDE into another PDE which is in a canonical form right. So, what is this canonical form we will discuss ahead. So, the idea is that you rewrite this differential equation in terms of two new variables zeta and eta, both zeta and eta are functions of x comma y right. So, we have the freedom to choose these two functions and indeed we must choose these you know variables in such a way that this Jacobean is non zero right.

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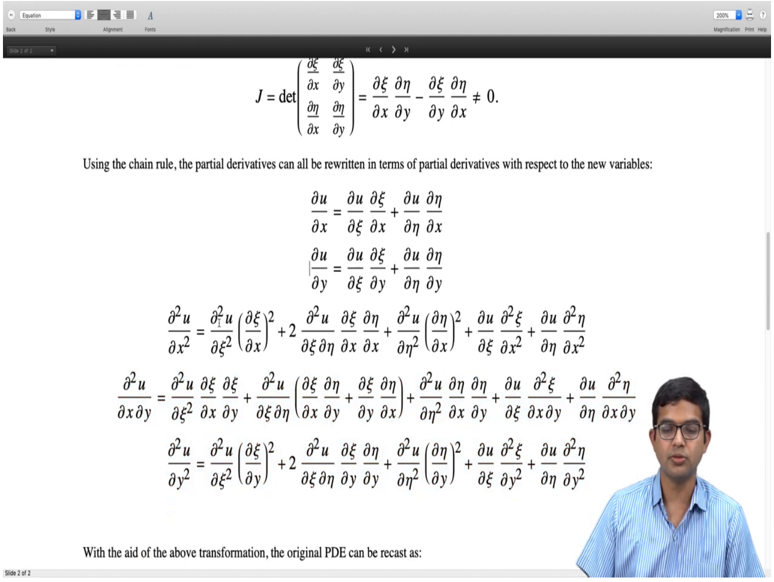

So, that basically means that this zeta and eta are independent you cannot take zeta and you know eta to be some factor of zeta for example, then it's not going you are not going to be able to compress information which has two independent variables in terms of just one independent variable that is not happening right.

So, you know this Jacobian being non zero ensures that zeta and eta are valid two new independent variables in terms of which a whole partial differential equation can be recast into a form which is more convenient to work with right. So, the idea is.

So, now it's a matter of cranking this machinery. So, there is this chain role using which we can write all these partial derivatives in terms of the new variables. So, dou u by dou x must be written in terms of dou u by dou zeta and then dou zeta by dou x plus dou u by dou dou eta times dou eta by dou x and similarly you will have a relation for dou u by dou y.

So, now, when we take the second order derivative dou squared u by dou x squared you know will get you dou squared u by dou z squared and so, then you get a dou z by dou x time dou z by dou x that is going to give you the square.

And then you know when you take a derivative with respect. So, dou squared u by dou z zeta dou eta, then you get a dou zeta by dou x times dou eta by dou x and likewise you will get a similar cross term from the other one and so, you get a factor of 2 and then of course, you have dou squared u by dou eta squared times dou eta by dou x times dou eta by dou x.

So, this is you know when the first three terms come about when you know when you take this derivative with respect to dou x considering you know dou u by dou zeta you are taking a derivative with respect to this part. But then there are also two more terms which come about where you treat this dou u by dou zeta is basically a constant.

So, when you are taking a derivative with respect to x, then you get a dou squared zeta by dou x squared and likewise you have a dou squared eta by dou x square eta. So, well I mean you have a similar relation you know slightly different you have to take care of these terms instead of a factor of 2 you get these two different terms right when you take a derivative with respect to dou x and dou y. So, it's like double derivative, but with both x and y in the you know with respect to which the derivatives are taken.

So, you can convince yourself that indeed this is the expression you get right. Its just a matter of taking this you know dou u by dou x and taking a derivative with respect to dou by dou y or equivalently you can take dou u by dou y and take a derivative with respect to dou xs both are going to give you the same answer and then dou squared u by dou y squared is going to be similar to dou squared u by dou x squared. So, in place of x you put an y. So, these are all very general. You know these are completely general results.

So, given some u you can take a derivative with respect to x and with respect to y and second order derivatives and so on and they all will be written in terms of these new variables now. So, what this will do is if you can plug in. So, wherever you have. So, you go back to this original PDE and then wherever you have dou squared u by dou x squared, you must rewrite it in terms of this expansion.

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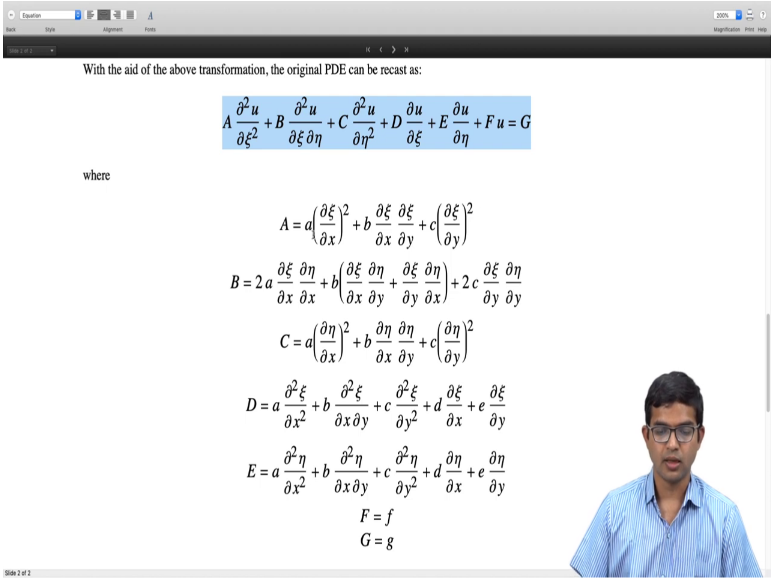

So, basically what it will do is, you can check this it will basically give back another partial differential equation in terms of zetas and etas right. So, it's going to look like this where these new coefficients are connected to the old coefficients in terms of these relations, I will allow you to check this.

So, for example, A is just A times dou zeta by dou x squared. So, we can look at this. So, if I look at dou squared u by dou zeta squared for example, it comes from here. So, there is one term like this dou zeta by dou x the whole squared. So, that also appears here.

So, because there is a factor of A and then whereas dou squared by dou zeta squared appears in this term here. So, then you have a dou zeta by dou x times dou zeta by dou y and indeed that appears here there is a factor of B associated with this term and then indeed there is also plus C times dou dou zeta by dou y the whole squared.

So, indeed. So, A you know where which is the factor that you tag along with dou squared u by dou zeta squared comes in these three different forms you can check this right and likewise we can work out B in terms of all these partial derivatives C in terms of these partial derivatives D again and E and F and G are basically these last two terms.

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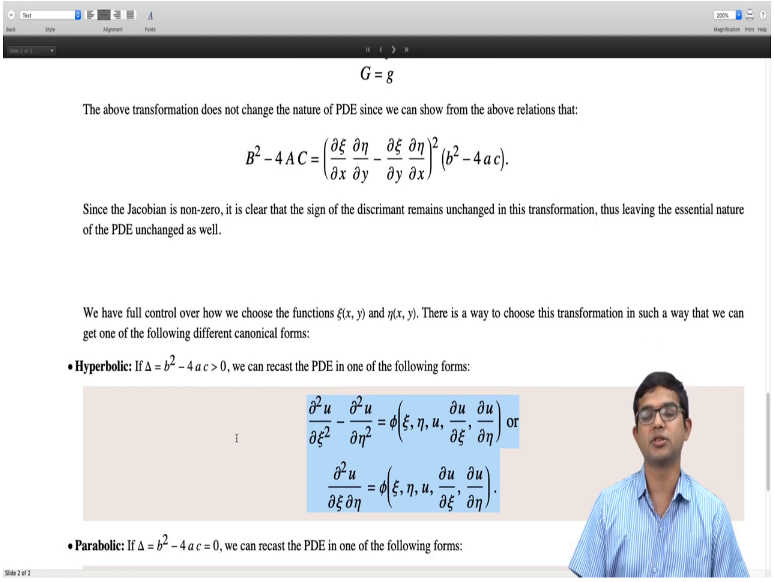

So, you see that you have you know f acting on u small f acting on u will become capital F acting on u and small z acting on z will become capital Z acting on u that is the stuff which appears on the right hand side. So, you can rewrite this PDE in terms of a new PDE and it's going to look like here and now. So, the key point is that because we can show that B squared minus 4 AC.

So, you have a partial differential equation again a second order partial differential equation you have a new a b and c. So, I said that it's only a b and c which count as far as the type is concerned and so, this transformation leaves this type invariant right. So, that we can see because it's possible to show. So, this is going to be an exercise. You must show using all these relations that the discriminant of this new partial differential equation B squared minus 4 AC capital B squared minus 4 times capital A times capital C is related to the old discriminant where this expression.

So, there is this factor squared times B squared minus 4 AC. Now since there is a square here which means that it's going to be a positive number. So, therefore, the sign of this is unchanged. If this is negative this is also going to be negative and if this is positive then this is also going to be positive and if B squared is equal to 4 AC of course, capital B squared minus 4 AC also is going to be 0.

Therefore, I mean that comes about because this Jacobian has been chosen to be non-zero. If this were 0 of course, then you may take a positive number and take it to be 0. So, the

Jacobian being non zero we said is a requirement. Because these new variables zeta and eta are independent therefore, the nonzero nature of the Jacobian ensures that the type of this differential equation is unchanged under this transformation.

Now based on which type of the equation is based on the value of this discriminant delta, you know if you go back and make this transformation right. So, there is full control over zeta and eta. So, we will go into these details in the lectures to follow. So, the point is that there is a way to choose this zeta and eta right. We have full control on zeta and eta. You can choose your zeta and eta such that if it is a hyperbolic differential equation you start with this complicated looking differential equation, but you can rewrite this in this partial form right.

Dou squared u by dou zeta squared minus dou squared u by dou eta squared is some function of all this stuff zeta eta u dou u by dou zeta whatever needs to come can come on the right hand side in terms of all these you know arguments can come into this function, but the key point is that the second order derivative with respect to u with of u with respect to zeta and of u with respect to eta the sign associated with these two terms are opposite or you can equally write this you know resulting differential equation as the second order derivative with respect to zeta and eta as some function right.

So, we will see an example and we will see exactly how this happens right. So, this is one form. I mean this is one possibility if it is a hyperbolic partial differential equation it can be rewritten in this manner from which it becomes you know it's simpler to handle this canonical form right.

Sometimes, it's possible to use a canonical form and immediately write down the solution and, but sometimes even with the canonical form there is still some more work to do, but the key point is that if you can bring it to the canonical form then we can develop techniques for the canonical form itself.

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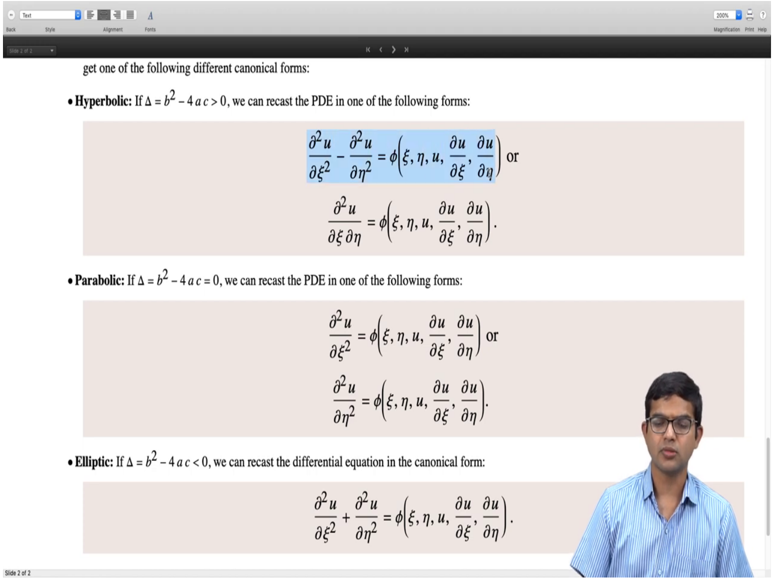

And so, that is where the power of this classification lies now if its of a parabolic kind. So, delta equal to b squared minus 4 ac we can recast this PDE in one of these forms dou squared u by dou zeta squared itself will be a function of the stuff equivalently you might be able to write it as dou squared u by dou eta squared it does not matter you know which variable its really these two are in some sense really the same because you can there is no hard and fast rule about which variable you call zeta and which you call eta.

So, there is a way to cast it either in this form or in this form and then there is the elliptic form. So, which is where the sign of the second order derivative with respect to zeta and the sign of the second order derivative with respect to eta are the same. So, dou squared u by dou zeta squared plus dou squared u by dou eta squared is some it can be a complicated function of zeta eta u dou u by dou zeta and dou u by dou eta right.

So, this depends on the specific problem at hand right. So, the key point is that you can take a partial differential equation of this generic type and there exists a transformation as we will see we will. So, explicitly how to do this which can bring it to you know one of these three canonical forms right.

So, there is some you know some details about how you can write it in this manner on or in this manner if its hyperbolic PDE and again in this manner or in this manner if it is a parabolic type and there is one sort of generic type if its an elliptic type of differential equation, but really you know there is a way to sort of think of these interchangeably.

So, that is why this is the canonical form for hyperbolic differential equations, this is for parabolic and this is for elliptic. So, we will go over this again in detail in the lectures ahead. That is all for this lecture.

Thank you.