Mathematical Methods 2
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Bessel Functions
Lecture - 58
Bessel functions: differential equation

So, we have looked at several properties of Bessel functions starting from the series definition. So, in this lecture, we will work out the differential equation you know from which Bessel functions really come out right.

So, the differential equation is often the starting point, but in our in the treatment that we have you know used, we are going to start from the we have already given the series definition and we will directly from the recurrence relation properties, we will show in this lecture how to work out the differential equation ok.

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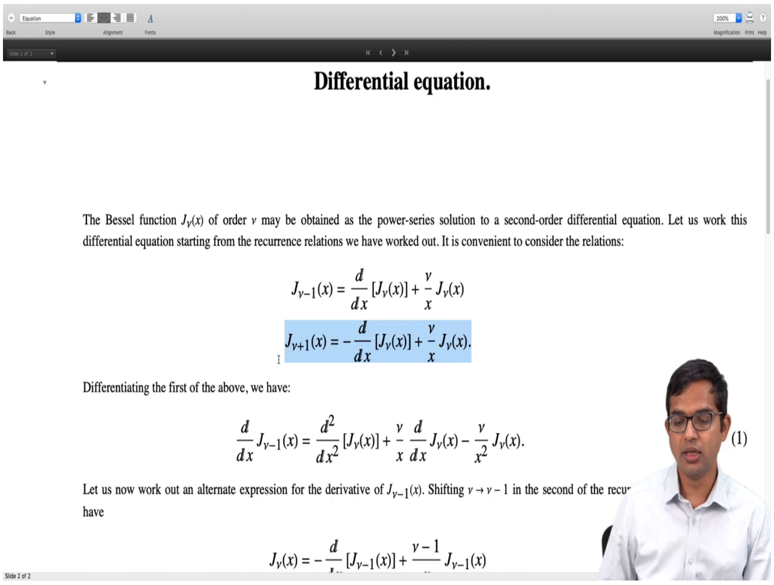

So the idea is that we write down these two recurrence relations, you know they are the recurrence relations which relate Bessel function of a certain order to the derivative of the Bessel function of the next order or the previous order right. So, J nu minus 1 of x is related to the derivative of J nu of x and then, there is also this extra term nu by x times J nu of x and J nu plus 1 of x is equal to minus the derivative of J nu of x and there is this correction plus nu by x J nu of x right.

So, we will use these two recurrence relations and work out the differential equation for which you know the Bessel function J nu of x of order nu is a solution right. So, let us take a derivative of the first of these two recurrence relations. So, we have d by d x of J nu minus 1 of x is equal to the second order derivative d squared by d x squared of J nu of x. Then, even if you take a derivative of this stuff, you get a plus nu over x times the derivative of J nu of x minus nu by x square times J nu of x right. So, that is what is here.

So, let us work out the left-hand side d by d x of J nu minus 1 of x in a different way and then, we will simply equate the right-hand sides which will give us the differential equation that we are after. So, in order to do this, we will shift nu to nu minus 1 in this recurrence relation.

So, we have J nu plus 1 of x on the left-hand side, but we want to use this relation, but you know shift nu to nu minus 1 so, the left-hand side will now become J nu of x is equal to minus d by d x of J nu minus 1 of x plus nu minus 1 divided by x times J nu minus 1 of x.

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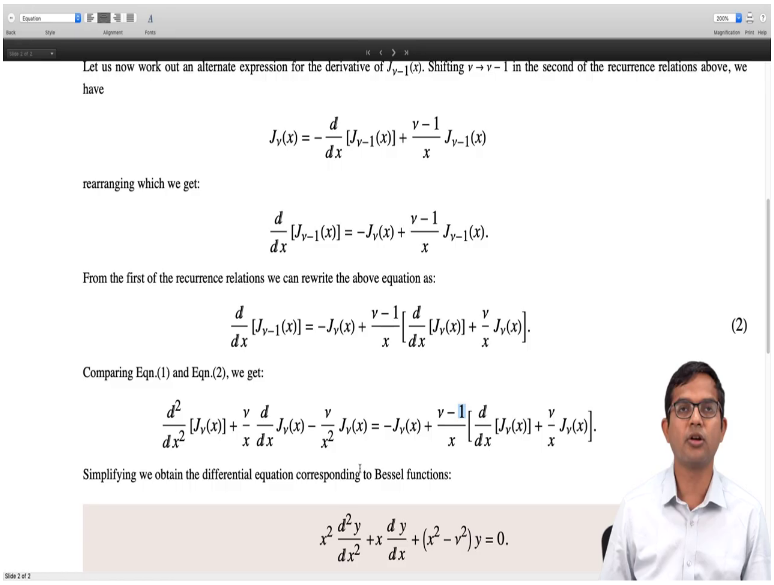

And so, if you rearrange this, we immediately have an expression for the derivative d by d x of J nu minus 1 of x is actually nothing, but so, this J nu can come to the right-hand side so, we have minus J nu of x plus nu minus 1 divided by x times J nu minus 1 of x. So, now, we have you know, but from the first recurrence relation so, it is we are still not completely done, I mean we are where after this left-hand side, but we can rewrite the right-hand side you know entirely in terms of J nu's right so, we do not want to work with J nu minus 1 of x like here.

So, we will rewrite J nu minus 1 of x using this recurrence relation in terms of this stuff which is all in terms of J nu of x. So, this relation can be rewritten as d by d x of J nu minus 1 of x is equal to minus J nu of x remains as it is, but in place of in the second term, we write this factor nu minus 1 over x as it is, but in place of J nu minus 1 of x, we write the entire stuff which comes really from the; from this recursion relation. So, we have d by d x J nu of x plus nu by x J nu of x.

So, now, we are done. So, we have actually equation 1 and equation 2 are both representations for the derivative of J nu minus 1 of x, all of this stuff is written in terms of J nu of x. So, there may be a derivative or a second order derivative, but there is nothing in terms of J nu plus 1 of x or in terms of J nu minus 1 of x.

So, we have managed to find two different expressions for the same quantity and so, we equate the right-hand sides of both of these equations, and we have this expression. So, d squared by d x squared J nu of x are will remain as it is plus nu by x d by d x of J nu of x minus nu by x square J nu of x must be equal to minus J nu of x, this also remains as it is, but we will have some cancellations from these terms nu minus 1 by x time d by d x of J nu of x plus nu by x J nu of x.

So, we see that this nu by x d by d x J nu of x will cancel with this nu by x, but there is minus 1 by x will remain and then, minus nu by x squared will also cancel with this stuff minus 1 times nu by x squared. So, we will be left with this minus J nu of x will come to the right-hand side, that will give us a plus J nu of x and then, we will also be left with this nu squared by x squared J nu of x.

So, in fact, this you know careful rearrangement of terms so that all the stuff appears on the left-hand side can be carried out and then, I will rewrite this resulting differential equation as a second order differential equation, it is the second order differential equation, but I will write it in terms of y and x just to highlight that. This is really a differential equation in its own right and this could have been the starting point of this discussion as it is the case in many other treatments.

So, what I have done here is, I mean I have multiplied throughout with x squared so, this is x squared d squared by d x squared in place of J nu of x, I write y, then I have plus x d y by d x so, this nu has gone and only this minus 1 by x is the only thing that remains and that gives me x d y by d x you can check this.

And then, you know the stuff involving nu squared by x squared and this one together they will amount to just x squared minus nu squared times y is equal to 0. So, this is like the defining you know differential equation of which the solution is the Bessel function of order nu.

And so, if we look at this as the starting point and so and look for a series solution where the Frobenius method right. So, we can get back the series expansion and then, we will see that that series expansion coincides with the series expansion definition that we started with and you know other properties can be worked out.

Now, it is a second order differential equation so, it must have a an independent alternate solution as well and so, in this case, if for you know generic nu not equal to an integer, the independent solution turns out to be straight forward, it is already there, I mean we see that this is there is a nu squared along which appears here. So, if J nu of x is a solution, then so is J minus nu of x right because it is you know in place of nu, you should put minus nu you get back the same differential equation.

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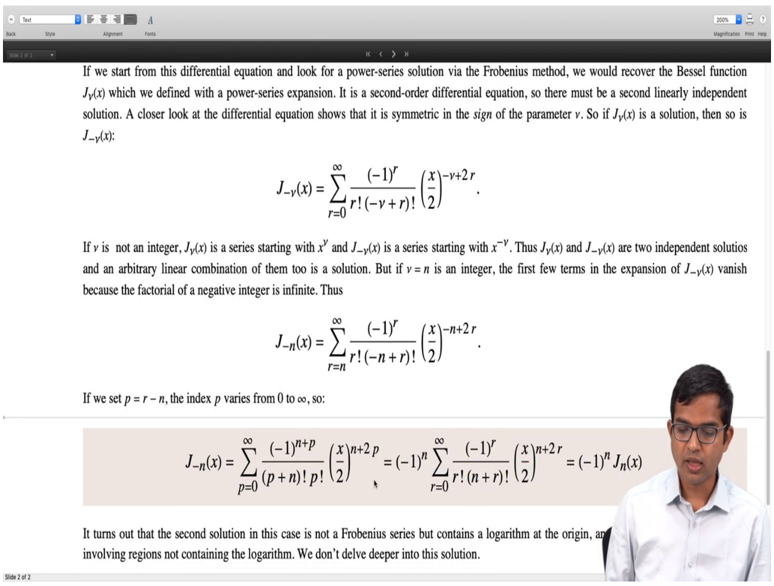

So, therefore, let us go to the series expansion and put minus nu in place of nu. So, we have J minus nu of x is equal to summation over r going from 0 to infinity minus 1 to the r divided by r factorial, I have a minus nu plus r the whole factorial in the denominator, x by 2 the whole power minus nu plus 2 r.

Now, if nu is not an integer in general, I mean we can immediately see from this relation that in this relation and the corresponding relation for J nu of x that J nu of x is a series which starts with the power x to the nu whereas, J minus nu of x is a series which starts with x to the minus nu.

So, we can immediately argue that basically J nu of x and J minus nu of x are not only two solutions of this differential equation, but in fact, they are independent solutions. So, therefore, any arbitrary linear combination of J nu of x and J minus nu of x is also a solution of this differential equation. So, that basically is the full general solution of this differential equation.

But on the other hand, if nu is equal to n, if it is an integer, then it turns out that J minus n of x is actually not an independent solution as we will see now right. So, if you have J minus n of x where you know n is an integer, then we have this expansion as it is, I have just plugged in its basically the same expression except that in place of nu, I have n.

Now, it turns out that the first few terms here will actually vanish. So, the reason is that we have minus n plus r the whole factorial in the denominator and the factorial of a negative integer is actually infinite. So, you have 1 over or some stuff over infinity so, basically all those terms do not contribute and so, in fact, here the series expansion actually starts from n.

So, the first you know terms starting from 0 all the way up to n minus 1, the first n terms you know if you just take this stuff and put n in place of nu, we can immediately argue that this is the same as r going from n to infinity, the reason is that you know when r is equal to 0, r equal to 1, r equal to 2 all the way up to r equal to n minus 1, this factorial is going to give us actually infinity right.

So, factorial function as we have seen is generalized using the gamma function and then, you can check that the gamma function you know the definition for factorial using the gamma function will actually give you infinity for the factorial of a negative number. So, all these terms, the first n terms are gone and so, then, you start from r equal to n to infinity.

And now, we derived this in terms of a change of variable. So, r minus n can be defined as p and then this index p will then from will run from 0 to infinity so, p goes from 0 to infinity minus 1 to the in place of r, we have to write n plus p and this r will becomes p plus n and this minus n plus r is of course, p.

So, in place of minus so, this should actually not be minus nu, but it should be minus n so, let us correct this. So, minus n plus 2 r and then, we have x by 2 to the whole power n plus 2 p right so, r is replaced by n plus p. So, plus 2 n will give you n, then this also 2 p.

Now, this is the same as I can pull out this minus 1 to the n outside and so, then I am left with so, I might as well use I mean it is a dummy index, I am just rewriting it in terms of r because its it gives back for us the familiar expansion we have for J n of x where we used r, it does not matter, we can use p or r, it does not matter. So, immediately we see that in fact, J minus of n of x is actually nothing, but minus 1 to the n times J n of x. So, J n of x and J minus n of x really carry the same information when n is integer.

So, therefore, ah the differential equation when nu is an integer, you know the Bessel function gives only one independent solution, the other independent solution turns out to have a logarithm. So, this Frobenius series does not quite work out and there is another solution, but we will not go into that.

So, these kinds of applications, you know these kinds of solutions of the Bessel function differential equation only appear in some very very special context and you know you do not want to work in intervals that include the origin because you have this you know if you have one this solution blowing up in a logarithmic fashion.

So, we are just pointing out in passing that indeed there is another solution even for when nu is an integer, but we will not go into the details of that solution ok. That is all for this lecture.

Thank you.