Mathematical Methods 2

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Bessel Functions
Lecture - 57
Recurrence relations

We can continue our discussion of Bessel Functions. In this lecture, we will look at some Recurrence relations which follow directly from the series representation of Bessel functions, and how a Bessel function of a certain order is related to the derivative of a you know of the Bessel function of an adjacent order ok.

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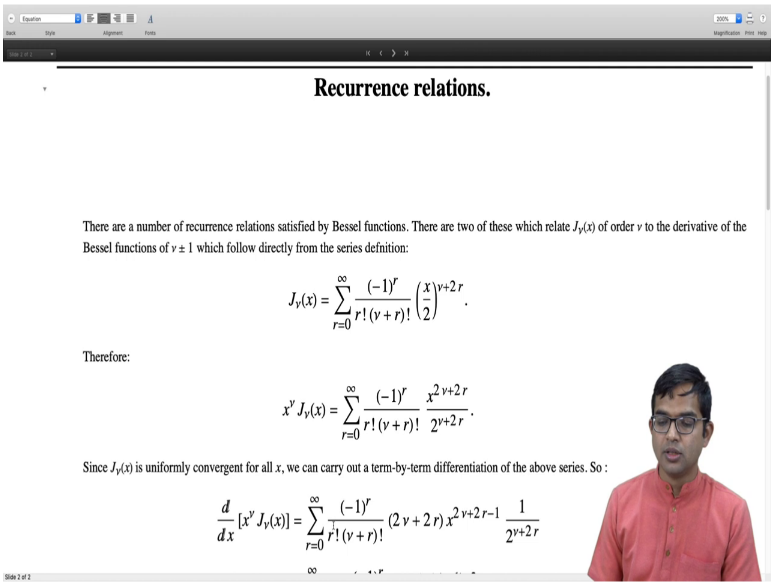

So, it turns out that you can relate J nu of x to the derivative of J nu plus 1 of x and the derivative of J nu minus 1 of x. So, and this can be derived directly from the series definition. And using this, we can also get some other useful recurrence relation. Basically it is the same two recurrence relations which when used in terms of certain convenient linear combinations give us other forms which have applications in a different context ok.

So, the starting point is of course this series definition of a Bessel function. And now if you take this function and multiply with x to the nu alright, so then we have you know the denominator here is 2 to the nu plus 2 r, but the numerator has become 2 nu plus 2 r everything else is basically the same.

And now we have said we did not show it explicitly, but it is true that this series is not only convergent, but it is actually uniformly convergent. So, if you want to take that derivative of this function, you can do it term by term. So, let us take the derivative of this function, and carry out term by term differentiation.

So, if I want to take the derivative of x to the nu times J nu of x, so it is going to be summation r going from 0 to infinity this stuff constant remains as it is. And so x to the 2 nu plus 2 r will give us 2 nu plus 2 r times x to the 2 nu plus 2 r minus 1, then we have this factor 1 over 2 to the nu plus 2 r as it is nu plus 2 r.

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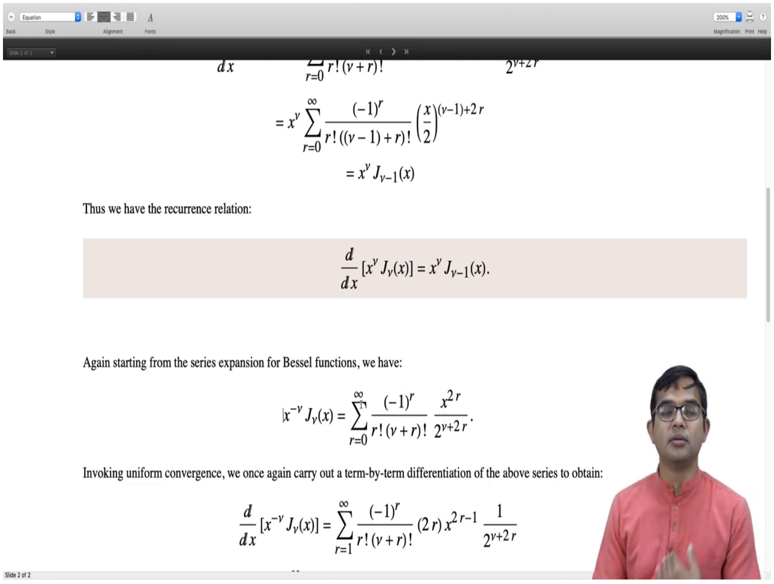

Now, we can pull out a factor x to the nu outside right. So, nu is independent of r. So, I can pull out x to the nu. And then when I look at this series, I have minus 1 to the r divided by r factorial, the denominator I have nu nu plus r the whole factorial which is nu plus r times nu plus r minus 1 the factorial whole factorial.

So, this nu plus r will cancel with this nu plus r, you know there is a 2 here which will cancel with this. And that 2 to the nu plus r, r will become 2 to the nu plus r minus 1 in the denominator. And I can combine these two and write it as x by 2 to the nu plus 2 r minus 1 which is more conveniently written here as nu minus r 1 plus 2 r. And once again the denominator here has become nu plus r minus 1 factorial which I am rewriting it as nu minus 1 plus r the whole factorial.

The reason I do this is because now this whole series is in suggestive form. All that has changed in comparison with this is that nu has become nu minus 1. So, immediately, we can read off from here that the derivative of this object x to the nu times J nu of x is the same as x to the nu times J nu minus 1 of x right.

So, we have the recurrence relation, it is useful to highlight this and write this as d by d x of x 2 the nu times J nu of x is equal x to the nu times J nu minus 1 of x. So, we have managed to connect J nu minus 1 of x with the derivative of the higher order Bessel function. So, it works in the other direction as well.

So, let us look at this again starting from the series expansion for Bessel functions. Suppose, we take J nu of x you know which we already have and multiply with x to the minus nu. So, now, in place of you know x 2 the nu plus r times x 2 the minus nu. So, this nu is gone.

So, we will be just left with x to the 2 r, and the denominator of course, we have to write it. So, we have this expansion for x to the minus nu times J nu of x is summation over r minus 1 to the r divided by r factorial nu plus r factorial times x to the 2 r divided by 2 to the nu plus 2 r.

Now, this once again we will take a derivative of this whole thing. And then again because of the uniform convergence, we can take a derivative, you know, perform the differentiation operation of the right hand side by just doing it term by term. So, all the stuff remains as it is. And then we get 2 r times x to the 2 r minus 1 and then you have to multiply with 1 divided by 2 to the nu plus 2 r.

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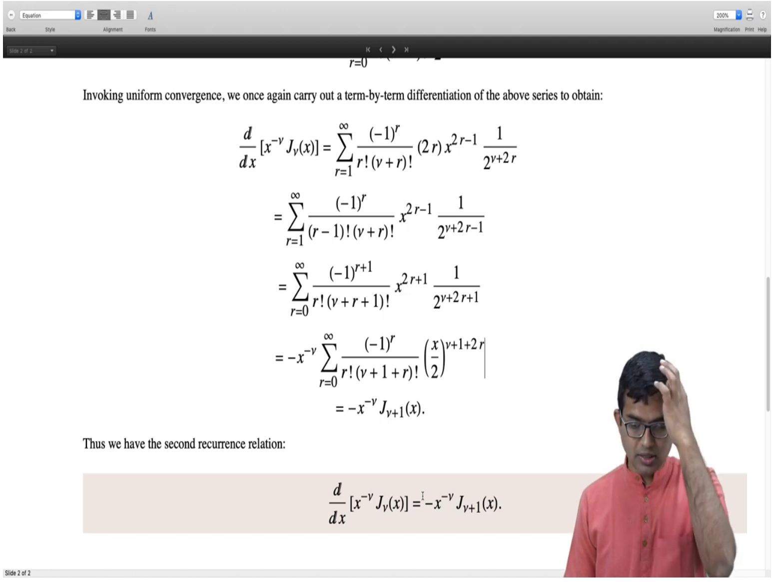

Now, we observe that you know all, although here the series started from r equal to 0, here you see that the series is actually going from r equal 1. The reason is simply that you know r equal to 0 is just the constant term that is gone. So, you can also see that if you put 2 r times x to the 2 r minus 1 and include r equal to 0, it is ok. And that will go this 2 times r will give you 0. So, in fact, it starts from 1.

So, I mean rewriting this whole thing we have minus 1 to the r minus. So, this r cancels with this r factorial in the denominator the denominator becomes r minus 1 factorial, then we have a nu plus r factorial. And then we have x to the 2 r minus 1 that this 2 which also cancels with one of these twos in the denominator, and we have 1 over 2 to the nu plus 2 r minus 1.

And now, in fact, it is convenient to introduce a new dummy variable right. So, r is being summed from 1 to infinity. Suppose, we introduce r minus 1 equal to s, so we would get minus 1 to the s plus 1 well, and then s would go from 0 to infinity. And in place of r minus 1 factorial, we will have s factorial in place of nu plus r the whole factorial, we will have nu plus s plus 1 factorial.

And r will become s plus 1 x to the 2 s plus 1. And then in place of the denominator we will have 2 to the nu plus 2 s plus 1. So, it is a dummy variable which is getting summed from 0 to infinity. So, I might as well just call it r. So, it is r going from 0 to infinity that is this expression I have here.

And now I can actually pull out one of these minus 1. So, I have minus 1 to the r plus 1. I want to write it as just minus 1 to the r. So, I pull out a minus sign, then I pull out this x to the minus nu. So, if I pull out x to the minus nu, I will have an x to the plus nu. So, this becomes x to the; x to the nu plus 1 plus 2 r, but also the denominator that is 2 to the nu plus 1 plus 2 r. So, I might as well write this as x by 2 the whole power nu plus 1 plus 2 r. And then we have you know all these factorial constants everything stays here.

And when we look at this expression, we immediately are able to connect it to the expansion for the Bessel function, but now it is for an expansion for a Bessel function of order nu plus 1. So, immediately we are able to write down this relation minus x to the minus nu times J nu plus 1.

So, collecting this, we have the second recurrence relation here. So, which is the derivative of x 2 the minus nu times J nu of x is equal to minus x to the minus nu times J nu plus 1 of x. So, we have managed to connect the derivative of a Bessel function with respect to the Bessel function of the next higher order.

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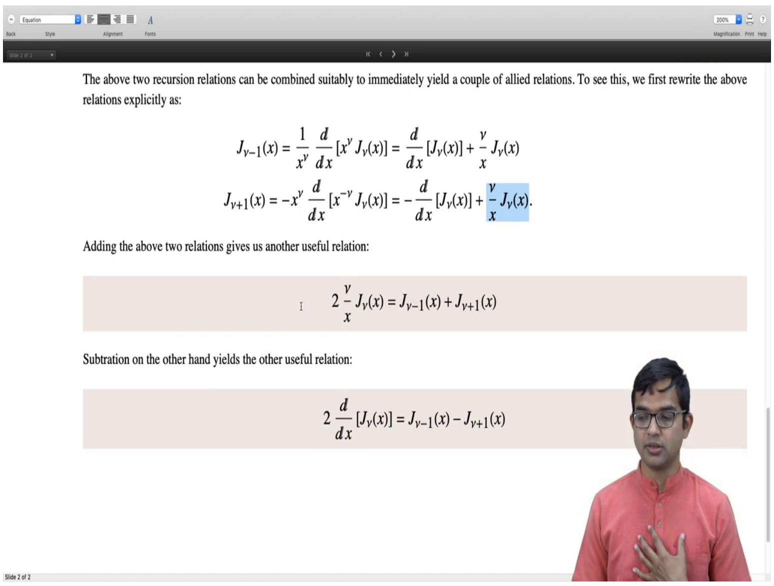

So, in fact, both these relations are you know can be written in this way right. So, what we have managed to show is that J nu minus 1 of x is 1 over x to the nu times derivative of x 2 the nu times J nu of x if I carry out this differentiation. So, I can either treat x to the nu as a constant and differentiate with respect to nu J nu of x.

So, then there will be a cancellation of x to the nu and nu x to the nu. So, I get d by d x J nu of x plus if I take a derivative with respect to x to the nu, then I will get nu times x to the nu minus 1 and that will cancel with x to the nu in the denominator, and I just have nu by x times J nu of x. So, this itself is a useful recurrence relation.

And likewise if I do the same with the other one and you know explicitly work out this differentiation, I have J nu plus 1 of x is equal to minus derivative of J nu of x plus nu by x J nu of x right. So, I have basically the same recurrence relations written in a slightly different way. And once again it is actually useful to rewrite this in the following way.

You can take the sum of these two or the difference of these two. If you take the sum of these two, you see that the first two term the first term in the first one and the derivative term in each of these will cancel, and so we are left with just 2 nu by x J nu of x is equal to the sum of these two Bessel function J nu minus 1 of x and J nu plus 1 of x is related to J nu of x. If you can just sum these two as going to be just 2 nu by x J nu of x, this is also of a useful recurrence relation to use.

And again if you take the difference it is the derivatives which you will survive if you take the difference and you know these terms the second term here and the second term here will vanish. When you are taking the difference and so we have the result 2 times the derivative of J nu of x is the same as J nu minus 1 of x minus J nu plus 1 of x. Ok, that is all for this lecture.

Thank you.