Mathematical Methods 2

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Orthogonal Polynomials
Lecture - 55
Laguerre polynomials: generating function

So in this lecture we look at the generating function associated with Laguerre polynomials. We will first of all argue for how you know what we claim is the generating function is indeed the generating function and then we will also look at an example of how the generating function can be used; in particular we will work out one of the recurrence relations associated with Laguerre polynomials, namely the three term recurrence relation, ok.

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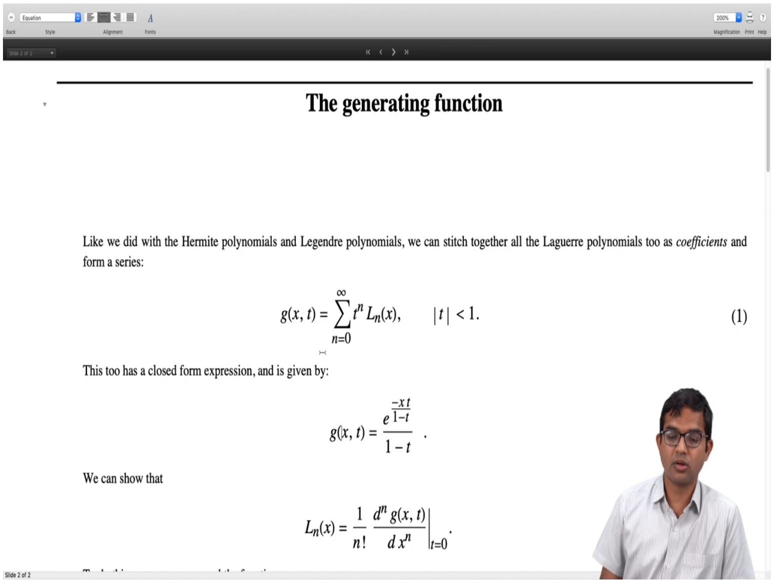

So, like we have seen before. So, the idea of the generating function is to find a function whose Taylor expansion you know contains these the coefficients of the Taylor expansion, which is valid of course in some region of convergence. Now, the coefficients are really the polynomials that are of interest, right. So, we think of a function which has you know two variables. The arguments of this function and so, the expansion is done in terms of the powers of in really a dummy variable.

We do not really care about this t and so, we are more interested in the coefficients which are actually functions of the other variable g of x comma t, right. So, for the, so like we did with hermi Hermite polynomials and Legendre polynomials; so you should recall how the convention was a little bit different when we were looking at the generating function for Hermite polynomials and with Legendre polynomials, right.

I mean you are consistent, if you stick consistently to a particular convention for a particular set of polynomials it is all good; but we might as well state that there can be some variation across polynomials. So, as far as Laguerre polynomials are concerned, it is convenient to consider this series defined like here g of x comma t is equal to t to the n times L n of x and n going from 0 to infinity.

So, there are no factors of n factorial none of that appear in this definition. So, it is just t to the n tagged with these coefficients which are the Laguerre polynomials. And so this series is known to be convergent when mod t less than 1, right. So, it turns out that this expression has a closed form, closed form which is simply given by e to the minus x t divided by 1 minus t the whole thing divided by 1 minus t, right.

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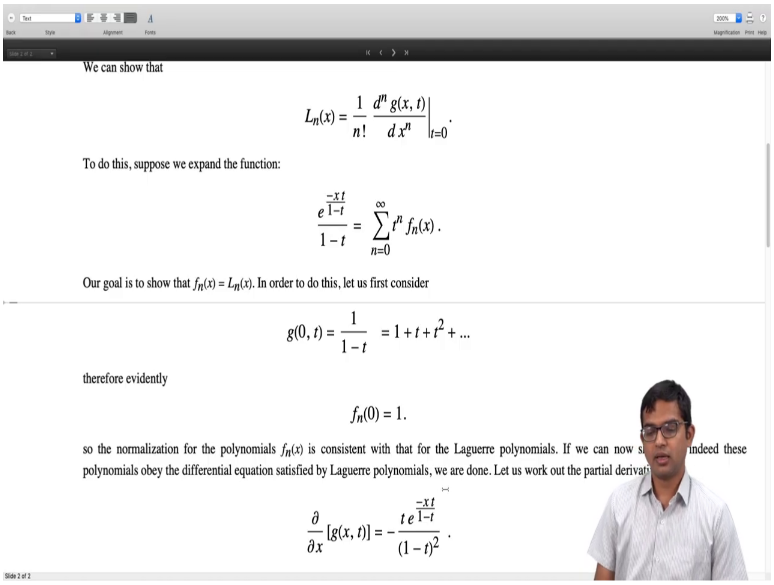

So, in order to show this what we have to really show is, if you take the nth derivative of this function with respect to x and then divide by n factorial and also put t equal to 0; you must show that the resulting polynomial is a is the Laguerre polynomial of order n, right. So, you know we will follow a method similar to what we did with Legendre polynomials; there are certainly other ways of doing this. So, our way is to you know say that, suppose we take this function and expand it, right.

So, we are going to get this kind of an x expression and I mean you are going to have factors of involving, you know powers of whatever stuff is here; you know you are taking an exponential of something and then, so you are going to get you know no power of this, but one power of this, two powers of this and so on. So, we are guaranteed that these coefficients are going to be functions of x which are really polynomials, because you are going to get increasingly higher powers of x.

So, it is guaranteed that these coefficients are going to be polynomials. So, the approach we will take is to first of all show the normalization holds and secondly, show that indeed this set of polynomials satisfy the differential equation corresponding to Laguerre polynomials. If you show these two, then it is basically like proving that indeed this is the generating function associated with Laguerre polynomials, right. The first part of it is straightforward, because we have to show that g of 0 comma t, right.

So, let us look at g of 0 comma t right, which is just 1 over 1 minus t; because you have put x equal to 0, so it is just 1 over 1 minus t, whose expansion we know is the, you know it is like the first infinite series expansion we become familiar with. And that is 1 plus t plus t squared all the way up to infinity and this is of course, convergent when mod t is less than 1. So, clearly you know the coefficients are all 1 here. So, indeed we see that f n of 0 is equal to 1, right.

So, we have managed to show that in this expansion, these coefficients are polynomials in x and the normalization of these polynomials are exactly like the normalization we want for Laguerre polynomials. So, all that remains is to show that, you know these polynomials are, they obey the differential equation corresponding to Laguerre polynomials and then we are done, ok. So, which we will do using the following technique, which is actually very similar to what we did with Legendre polynomials.

So, let us give some more detail here. And so, the idea is you know to take this function and try to differentiate it with respect to x partial derivatives, first order, second order with respect to x and then somehow couple it with the partial derivative of g of x comma d with respect to t. So, it turns out with a little bit of algebraic manipulation; we can show that there is a general constraint that these first order and second order derivatives with respect to x and this convenient, conveniently chosen term involving the first derivative with respect to t, partial derivative with respect to t. They all can be combined together in a nice linear combination to put them to show that they all add up to 0.

Let us let us do this. So, suppose we look at the partial derivative of g of x comma t with respect to x; so we can immediately see that, you will get you know this factor of minus t over 1 minus t comes out. So, we can simply write it as minus t divided by 1 minus t the whole square times e to the minus x t divided by 1 minus t. If you take another partial derivative with respect to x; you are going to get one more factor of this, minus t times t divided by 1 minus t.

So, together you can write this as plus t squared times e to the minus x t divided by 1 minus t the whole thing divided by 1 minus t the whole cube.

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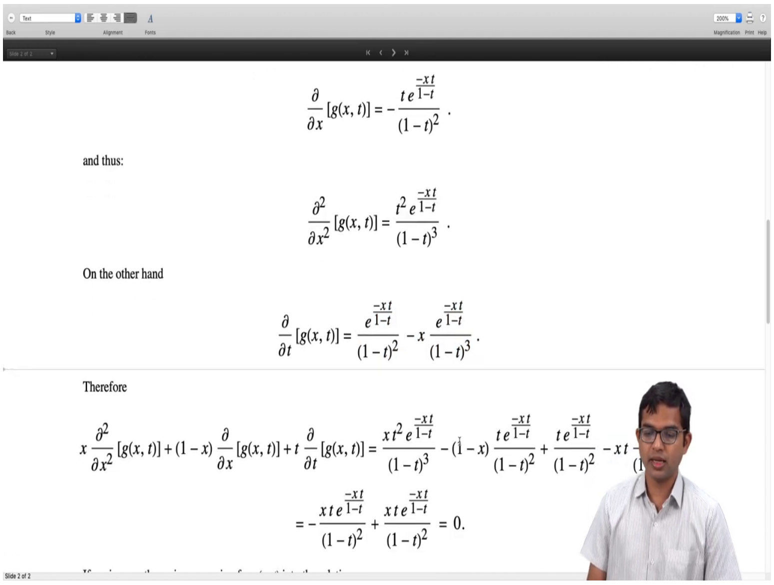

Now, on the other hand if you take a derivative, partial derivative with respect to t; it is a little more complicated, but it is still straight forward. So, what you do is, you treat. So, you have 1 minus 1 over 1 minus t, so you can take a derivative with respect to that, certainly this is a constant first. And so, you get minus 1 over 1 minus t the whole squared times minus 1.

So, that is the first time it is just 1; 1 over 1 minus t whole square times this. Then we have a you know you have to take a derivative with respect to the numerator and so, the denominator will remain as it is. So, you can carry out this algebra. So, I did it and I managed to combine some terms. And so, effectively all come down to minus x times e to the minus x t divided by 1 minus t, the whole thing divided by 1 minus t the whole power 3.

So, you should check that I have done the algebra correctly, but I think the answer is correct. So, indeed, this is going to be the partial derivative of this function with respect to t. Now, we will take this combination of you know these three terms; so there is a way to combine these three terms multiplying by suitable factors, which will go to zero. So, let us look at this quantity right. So, x times the second derivative.

So, if I take x and multiply with this; then I have, then I add this to 1 minus x times the first quantity and then I add t times this quantity.

So, if I do this, the first term will just become x times t squared times e to the minus x t divided by 1 minus the whole power, whole divided by 1 minus t the whole cube. Then I have a plus 1, 1 minus x times will become minus 1 minus x times this object here, which is just t times e to the minus x t the whole thing divided by 1 minus t and then the whole thing must be divided by 1 minus t the whole square. So, this is the second term and then the third term actually contains two terms.

So, I have to multiply by t. So, I get t times e to the minus x t divided by 1 minus t the whole thing divided by 1 minus the whole squared minus x times t times e to the minus x t divided by 1 minus t the whole thing divided by 1 minus t the whole cube. So, now, what I can do is, I can actually combine this term and this term; both of them have e to the minus x t divided by 1 minus t and then in the denominator you have 1 minus t whole cube and there is also this x t which is common.

So, if I pull out x t times this whole stuff common; then I will have times t minus 1, which will cancel with one of these 1 minus t whole cube. And so, basically I get minus x t times e to the minus x t divided by n minus t the whole divided by 1 minus t whole square. And then I can combine these two. So, each of them you see has just this stuff which is common. So, I have 1 minus 1 plus x, so that is just x t plus x t times this whole stuff. So, now, you see that this term and this term cancel and so, you just are left with 0.

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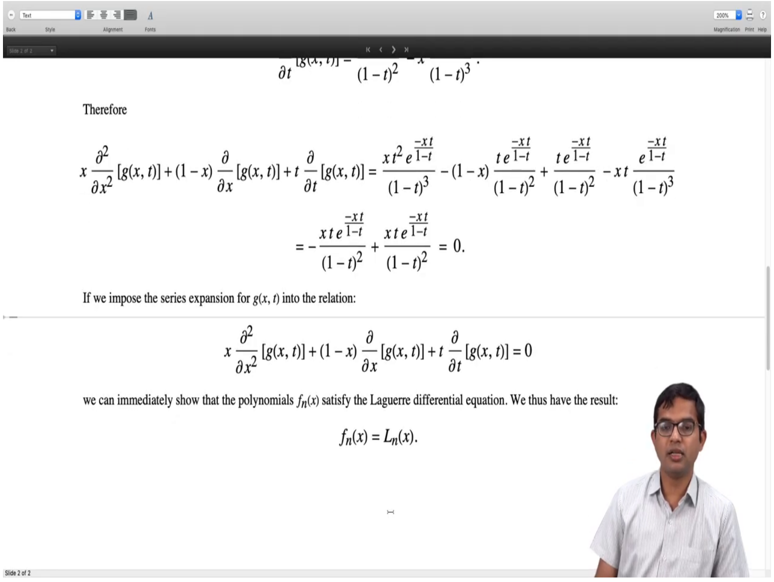

So, what we have managed to show is, we have this relation which is satisfied by g of x t; just purely from the form of this function g of x comma t by taking these derivatives, we have directly shown from what g of x comma t is that this relation must hold. Now, if g of x t satisfies this, then so does the expansion.

So, and that is how we will when we will impose that upon the expansion and therefore, we will see that this causes these f n of x to satisfy the Laguerre differential equation, right. So, I will leave the details for you to work out.

Because it is just a matter of some bookkeeping; all you have to do is you know plug in this relation, plug in this relation. Well, I mean this relation, plug in this relation in place of g of x comma t and then collect all the terms which are you know the coefficient corresponding to t to the n.

So, if you have a summation some coefficient t to the n must is going to be 0 for all values of t; then indeed it must each of the coefficients separately must be 0 and that will give you know the condition that f n of x indeed satisfy the Laguerre differential equation.

So, I will allow you to work that out and complete it, that is going to be homework. But I will show you some details of how you know this can be exploited and therefore, the technique that follows ahead is also a hint of how to complete this exercise.

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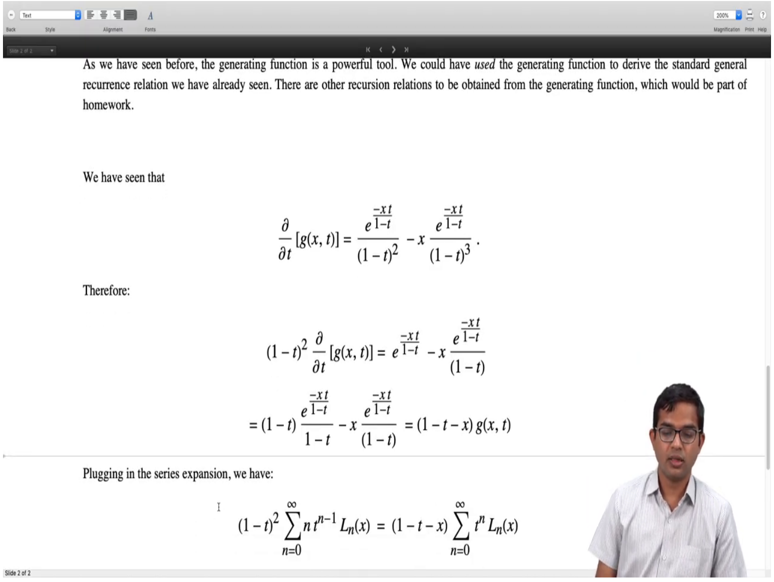

So, we will derive the three term recursion relation, you know which we already derived directly from first principles; but we can also use the generating function to work out the three term recurrence relation.

So, the way to do that is the following. So, we have seen that the partial derivative of g of x comma t with respect to t is e to the minus x t divided by 1 minus t the whole thing divided by 1 minus t the whole squared minus x times e to the minus x t divided by 1 minus t the whole thing divided by 1 minus t whole cube, right. So, basically we let us assume that we have managed to show that these coefficients are indeed the Laguerre polynomials. We have already accepted that that g of x command is the generating function for it, right. I mean basically the argument is like what we just said, right.

So, we managed to show that the differential equation is satisfied that these coefficients are polynomials and the normalization is good. So, that is the proof for this being the generating function. So, now, let us say we are going to use this result, this is the generating function.

So, this is the derivative and let us multiply throughout with 1 minus t the whole squared. So, if I multiply by 1 minus t the whole squared; then I get e to the minus x t divided by 1 minus t the first term and then I get minus x times e to the minus x t divided by 1 minus t the whole thing divided by 1 minus t.

There are two terms; then we want to bring it to the form, so that the right hand side also can be written in terms of g of x comma t. So, I multiply and divide the first term with 1 minus t. So, I have 1 minus t times the; this is really the generating function, then I have minus x times the generating function. So, I can collect all this and write it as 1 minus t minus x the whole thing multiplied by the generating function, which is g of x comma t. So, I have this relation 1 minus t the whole squared times the first partial derivative of g of x comma t with respect to t is equal to 1 minus t minus x times g of x comma t.

Now, I will bring in the series expansion and plug it into this relation. So, I have and then it is a matter of bookkeeping. So, I see I write this as 1 minus t the whole squared times; when I take the derivative, it is going to be n times t to the n minus 1 times L n of x is equal to 1 minus t minus x times g of x comma t.

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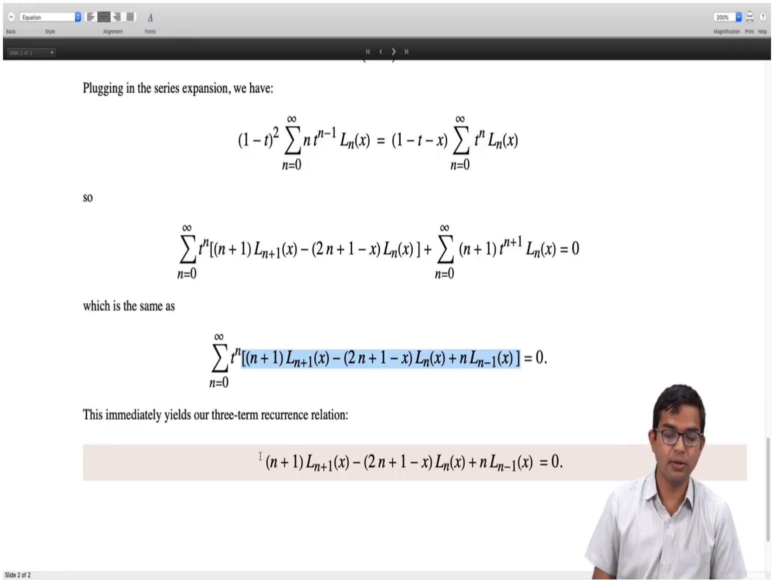

Now, there is a way to simplify this, right. So, you look at this term. So, I have 1 minus t the whole square can be written as 1 minus 2 t plus t squared. So, when I write 1 times this, it is the same as; see you have this term which instead of writing it as n times t to the n minus 1, I can make this transformation, where I send n minus 1 to k.

So, then this will become k plus 1 and then this will also become k plus 1 and k will run from, so k is equal to n minus 1. So, it is going to run from minus 1. But the first term is irrelevant; so in fact instead of running this from n equal to 0, I can as well run this from n equal to 1. And then basically I can rewrite this as in place of k, I can put back n.

So, that is going to be this first term. So, in place of n, I have n plus 1; in place of L n L n of x, I have L n plus 1 of x and then this will be multiplied by t to the n. So, that is the first term. Then I will have a similar result, now with 2 2 t minus 2 t and that 2 t is going to go with you know this n minus 1 and get back to t to the n itself. So, that I will keep it as it is; so I have minus 2 n you know that is the second term, so I have an L n of x.

But then I also have this 1 minus x on the right hand side, which is associated with t to the n, that also I can bring to the left hand side. So, I have minus 2 n plus 1 minus x L n of x. So, all of this stuff goes directly with t to the n. And then there is one more term here plus t squared; t squared will go with t to the n minus 1 and that becomes t to the n plus 1, right. And again on the right hand side I have a minus t times t to the n, that will become t to the n plus 1.

So, that these two I will collect it and write it as. So, I have a here you know plus n plus 1. So, if I bring t to the n plus 1. So, from this term with a minus sign, I bring it to the left hand side. So, that is going to become; well I mean not minus x, but the minus t, minus t times this stuff when I bring it to the right hand side, that is going to give me a plus L n of x. And now here I have a n plus, so I have a t squared, t squared times n times L n of x.

So, I have an n and a plus 1, so that is going to become n plus 1, right. So, there is, so there is this term involving summation with respect to t to the n; but there is also this term involving summation with respect to t to the n plus 1.

So, now, I can rewrite this second term also as a summation involving t to the n. So, I make the transformation n plus 1 equal to k. So, I have k times t to the k. So, n is equal to k minus 1. So, I get L k minus 1 and then if n is 0, then k will be 1. So, it will be k times t to the k L k minus 1, k starting from 1; but then since I have this factor with k, I might as well start it from 0, it does not matter.

So, I can still retain the same you know starting point k equal to 0, because I have this factor with k equal to 0 sitting here. And I might as well rewrite it as in place of k, I can put back in n; because it is a dummy variable which gets summed over. So, basically I am saying that if this whole stuff will just give me another n times L n minus of 1 of x all of them tagged along with this factor t to the n, n can still go from 0 to infinity, this whole thing must be equal to 0.

So, if you are not entirely convinced with the way I argued; so all you have to do is rewrite n plus 1 is equal to k and then convince yourself that indeed you will get this term. And also convince yourself that it is completely legitimate to keep the limits at you know k equal to 0 instead of k equal to 1 right; because that is the first term is just going to be 0 in any case. So, what it does is, it allows me to immediately extract this three term recurrence relation.

So, this whole stuff must be equal to 0 for any value of t; therefore every one of these coefficients must be 0 and so, we have this three term recurrence relation. We already derived this from an in an independent approach. So, we have the result at n plus 1 times L n plus 1 of x minus 2 n plus 1 minus x the whole times L n of x plus n times L n minus 1 of x is equal to 0, ok.

Thank you.