Mathematical Methods 2
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Orthogonal Polynomials
Lecture - 53
Laguerre polynomials: three-term recurrence relation

Ok, so in this lecture, we show how to obtain the three-term recurrence relation corresponding to Laguerre polynomials starting from first principles.

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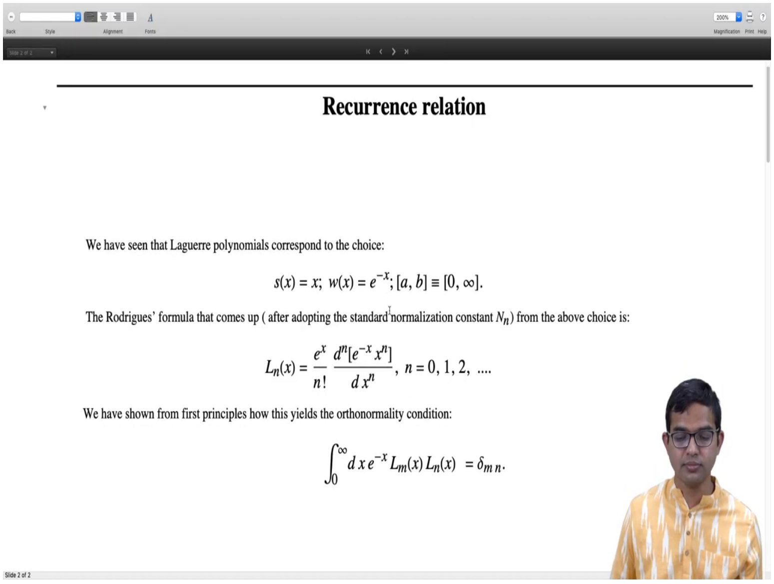

So, Laguerre polynomials are obtained when we set s of x to be x, w of x the weight function to be e to the minus x, and the interval corresponding to Laguerre polynomials is from 0 to infinity. We have seen this. And the Rodrigues' formula corresponding to Laguerre polynomials is e to the x divided by n factorial times nth derivative of the product e to the minus x times x to the n.

And so this will give us Laguerre polynomials for all integer values starting from 0, 0, 1, 2 all the way up to infinity, you can keep on increasing them. And we have also seen how we have this orthonormality condition integral 0 to infinity d x into the minus x L m of x times L n of x is equal to just the chronicle delta m n.

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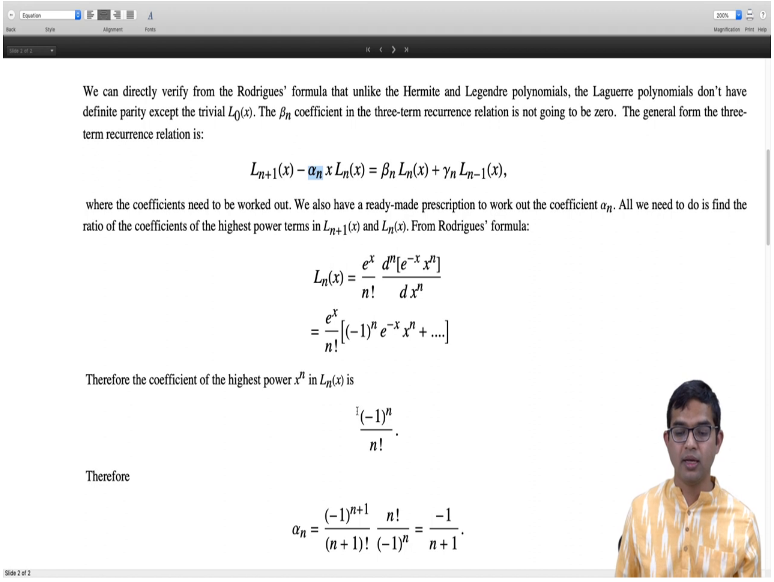

Now, in this lecture, we will work out the three term recurrence relation. So, the general three term recurrence relation is of this form right. So, L n plus 1 of x minus some coefficient alpha n times x times L n of x is equal to beta n times L n of x plus gamma n times L n minus 1 of x, right.

So, in the case of Hermite polynomials and in the case of Legendre polynomials, we argued that this coefficient beta n is going to go to 0 because of their, you know, well-defined parity for each of the polynomials. And also the fact that these disparity would oscillate, right.

So you would get either even polynomials or odd polynomials depending on the you know the term that you are considering, the member of the sequence, right. If you started with 0, you would get an even polynomial; n equal 1 would give you odd, and again back to even and so on.

But we have seen that Laguerre polynomials do not have definite parity. So, this argument does not hold anymore. And therefore, we have a full genuine three-term recurrence relation. So, we will have to work out all these three coefficients alpha n, beta n and gamma n which is what we will do in this lecture, ok.

So, we start with Rodrigues' formula as usual L n of x is equal to e to the x divided n factorial times nth derivative of this product. And then we quickly write down the highest order term in this because it's going to be useful in a moment as you see. So, the highest order term is obtained when all these derivatives are taken only with respect to e to the minus x.

So, x to the n is left as it is. So, it is a nth degree polynomial. So, the highest order term is going to be when you just have minus 1 to the n, and divided by n factorial basically, right. So, the highest order terms coefficient is minus 1 to the n divided by n factorial. We have already seen this when we were working out the normalization integral.

Now, from a few lectures ago, you will recall when we were discussing orthogonal polynomials from a general perspective, the way to get this coefficient alpha n is to take the coefficient of the highest order term in L n plus 1 of x, and then divided by the corresponding coefficient the highest order coefficient in L n of x; so which we are ready to do.

So, alpha n is simply minus 1 to the n plus 1 divided by n plus 1 factorial divided by minus 1 to the n divided by n factorial which we can write it in this manner. And so we have cancellations which happen here. So, n plus 1 is the only thing which will survive because you can write the denominator as n plus 1 times n factorial, and n factorial cancels. And together this minus 1 to the n plus 1 divided by minus 1 to the n is of course, just minus 1. So, alpha n is immediately seen to be simply minus 1 over n plus 1.

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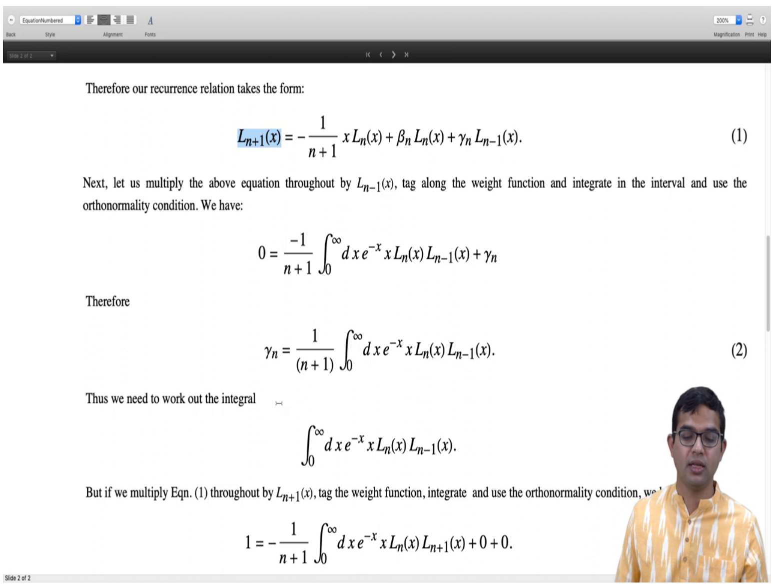

Therefore our recurrence relation also already has this you know form which is becoming more and more concrete. So, L n plus 1 of x is equal to minus 1 over n plus 1 times x times L n of x plus beta n times L n of x plus gamma n of times L n minus 1 of x. So now, there are these two coefficients beta n and gamma n which need to be worked out. And to do this, we will exploit the orthonormality properties of the Laguerre polynomials.

So, what we do is, we multiply throughout with L n minus 1 tag the weight function along and integrate. So, when we do this, the left hand side you know L n minus 1 and L n plus 1 are by design orthogonal to each other. So, when you do this integral on the left hand side, it is going to be 0. And then we have this integral minus 1 over n plus 1 time 0 to infinity x times; well, I mean d x e to the minus x times x times L n of x times L n minus 1 of x.

Plus again, we can argue that L n and L minus 1 are orthogonal to each other, so the second term also will vanish. So, only the third term which will, which will stay. So, we have, but the third term is again something which we already know. So, the normalization integral is just 1 when L n minus 1 appears 2 times, so that the third term can be simply written as gamma n times 1 is just gamma n right.

So, we immediately have an expression for gamma n right. So, we can bring this gamma n to the left hand; to the left hand side. And I mean just take care of the signs properly. And so we have one over n plus 1 times integral 0 to infinity d x e to the minus x x times L n of x times L n minus 1 of x, right.

So, although, we have this you know gamma n in terms of an integral, we need to also work out what this integral is right. So, we write down an answer for it. And so let us do this integral again in a clever way. So, to do this what we will do is we will first multiply equation 1 throughout. Instead of multiplying by L n minus 1, let us multiply by L n plus 1 of x. And you will see that we will be able to extract this integral.

So, again we tag the weight function and do the integral from 0 to infinity. So, on the left hand side, it is a normalization integral so it just goes to 1. And then we have minus 1 over n plus 1 integral 0 to infinity d x e to the minus x times x times L n of x times L n plus 1 of x. And then again we argue that beta the second term is going to go to 0 because L n and L n plus 1 are orthogonal to each other. And once again the third term also will go to 0 because L n minus 1 and L n plus 1 r orthogonal.

So, we are left with just one integral. And that integral is actually nothing but the integral we are after except that we have to see that here we have L n of x times L n plus 1 of x, whereas here we wanted L n times L n minus 1 of x. So, there is a ready fix for this and that is to just shift you know n r to n minus 1. If you change n time minus 1, so this term will become n minus 1, this will become L n. And so this is going to become n here instead of minus 1 over n plus 1 you will have minus 1 over n.

And then you bring it to the you know to the right hand side if you wish and you rearrange these terms in such a way that you write down this integral 0 to infinity d x e to the minus x times L n of x times L n minus 1 of x as just simply minus n. So, this is nothing but when along with equation 2, we can immediately say that gamma n is minus n divided by n plus 1, right. So, this means that our recursion relation has become even more concrete.

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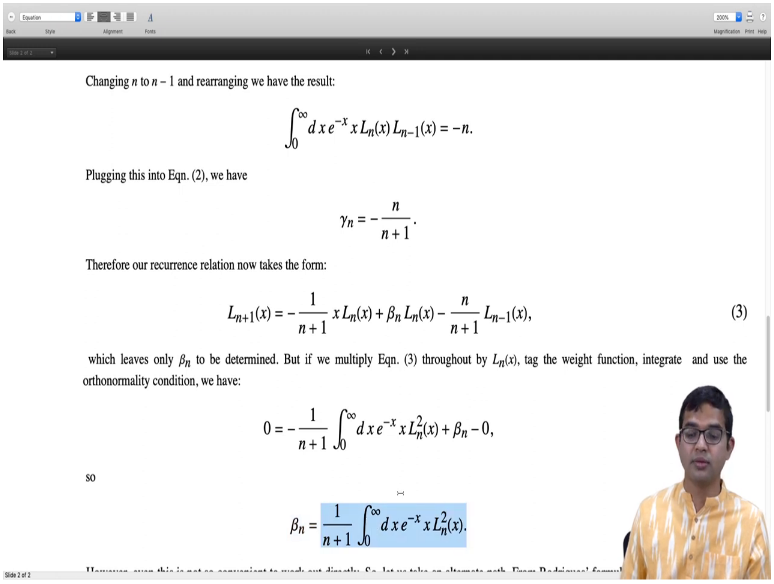

Now, we have two of these unknown coefficients out of the three. So, we have L n plus 1 of x is minus 1 over n plus 1 time x times L n of x plus beta n times L n of x minus n over n plus 1 times L n minus 1 of x.

So, only beta n remains to be worked out. And so we might be tempted to multiply throughout by L n of with L n of x and see if it can tell us the answer. So, of course, the left hand side will go to 0 because L n and L n plus 1 are orthogonal, and then we will be left with this term. So, L n time L n so you get an L n squared.

So, there is this integral which we will have to work out. Beta n times L, this of course we will just go to 1. And again by orthono orthogonality you can argue that this term also goes to 0, but the difficulty is we do not know how to work out these integrals. This requires work.

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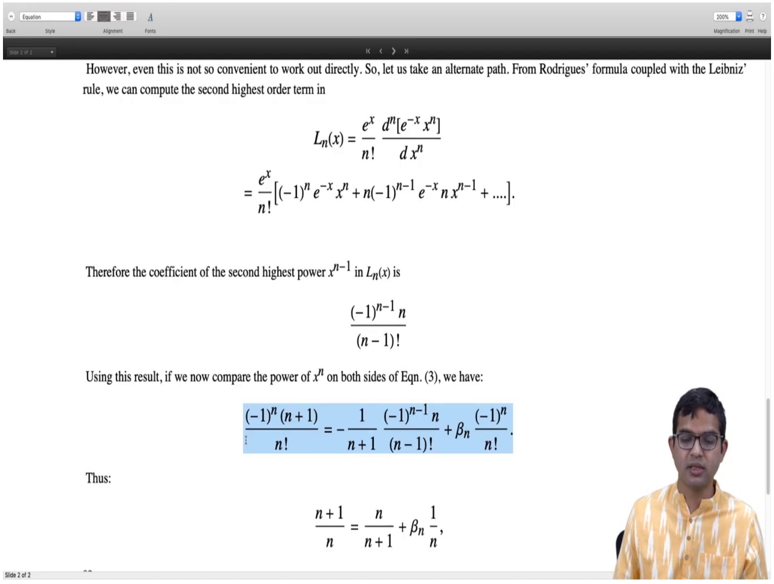

So, it turns out that in fact it is convenient to work out this coefficient beta n directly. We will start with Rodrigues' formula, and then use the Leibniz rule. So, Rodrigues' formula tells us you know L n of x is e to x divided by n factorial time this nth derivative of this product.

Now, let us work out not just the highest order term, but let us also work out the next order term. The highest order term after all comes when you take all these derivatives with respect to e to the minus x, so that is how you get a minus 1 to the n and then e to the minus x and x to the n remains as it is.

But then the next order term will come in when you take one derivative with respect to the second of this x to the n. So, that is going to give you n times x to the n minus 1, but there are you know you are doing this n times you could have chosen to do this derivative with respect to x in n different ways, so that is how we get this you know factor n.

So, or you can directly apply Leibniz rule and say that you should get n times the derivative the minus 1-th derivative of the first term times the first derivative of this second. So, we have n times minus 1 to the n minus 1 time e to the minus x that is the n minus 1-th derivative of this function. And then the first derivative of this function is n times x to the n minus 1 right. And then there are you know lower order terms after that which we will not consider now.

So, the coefficient of the second highest power x to the n minus 1 is actually nothing but we if you look at this the second term is minus 1 to the n minus 1 times n, one of these ns will cancel with the highest output power here, and you have a n minus 1 factorial in the denominator. So, this is the second highest power x to the n minus 1 in L n of x.

So, now what we will do is, we will go to equation 3 which is I mean this is the unknown which we want to work out beta n, we do not know. And so what we will do is we will compare the coefficient corresponding to the second highest order term on the left hand side and the same coefficient on the right hand side.

So, if you have L n plus 1 of x, it is a polynomial of degree n plus 1. So, let us look at the coefficient corresponding to x to the n. So, according to this prescription, it must be minus 1 to the n times n plus 1 divided by n factorial right. So, we have L n plus 1 of x and we are looking at the nth term, so that is what this is.

And then again we have you know x times L n of x right. So, we L n of x is a polynomial of degree n. So, if you take the highest you know power here and multiply with x that is going to correspond x to n plus 1, but that is not what we want. So, we want the next order term, and that order term is exactly what we just calculated. So, it is just exactly minus 1 to the n minus 1 times n divided by n minus 1 factorial, then we have to tag along this factor minus 1 over n plus 1 because we have this factor minus 1 over n plus 1.

And then we also have plus beta n times you know this is the highest order term. So, L n is a polynomial of degree n. So, we are interested in the coefficient corresponding to x to the n, so that is the highest order term in here which we already worked out, and so that is nothing but beta n times minus 1 to the n by n factorial.

And the other term does not contribute because this is after all n it is a polynomial of degree n minus 1, and we are interested in comparing coefficients corresponding to x to the n. So, this would not contribute. And all we have to do is now do some simplification here. It is just some algebra and we can immediately show I mean all this n minus 1 factorial will just cancel and you're left with n here on the left hand side.

So, n plus 1 divided by n minus 1 to the n will go and then you have minus times minus 1 to the n minus 1. So, again minus 1 to the n goes throughout. So, you have n plus 1 divided by n is equal to n over n plus 1 plus beta n divided by n.

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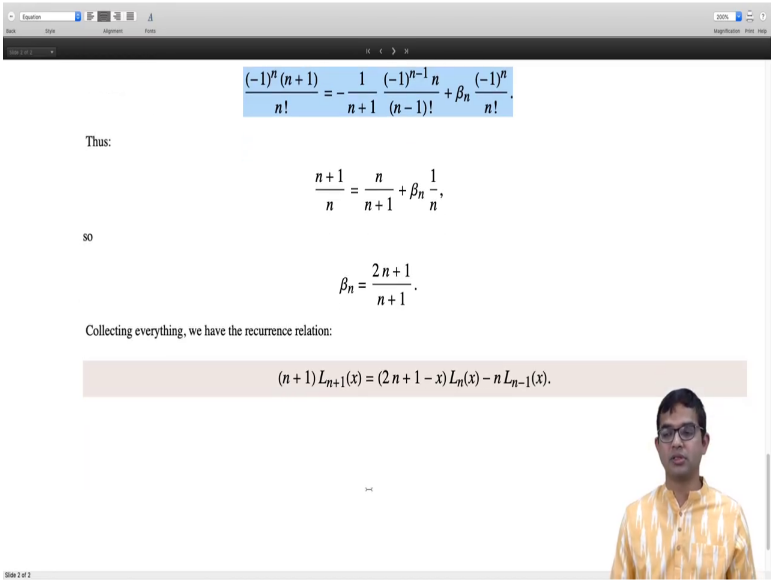

So, n minus 1 factorial has been pulled out throughout. And immediately we can write down beta n as 2 n plus 1 divided by n plus 1. You can check this. It is a very simple algebra, and that is basically it we just have to collect all these three pieces of information, and then write down the three-term recurrence relation. So, it turns out the three-term recurrence relation here is n plus 1 times L n plus 1 of x is equal 2 n plus 1 minus x times L n of x minus n times L n minus 1 of x, ok. So, that is all for this lecture.

Thank you.