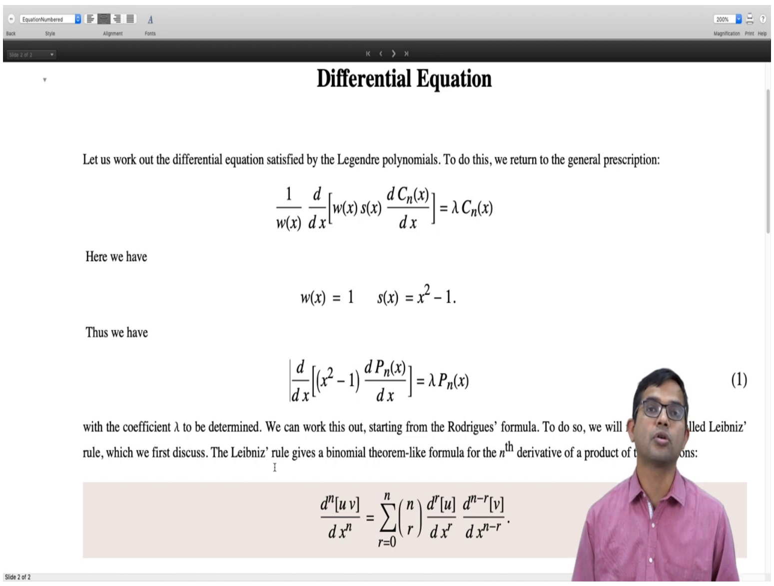
Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Orthogonal Polynomials
Lecture - 50
The differential equation corresponding to Legendre polynomials

So, in this lecture, we are going to work out the differential equation corresponding to Legendre polynomials. So, like we said in our development of you know the theory of orthogonal polynomials, we started with this abstract formulation worked out the polynomials, and then we have you know we have treated the differential equation corresponding to the polynomial as sort of a property of the polynomial right, but many times it is the treatment is carried out in the opposite direction.

So, you start with the differential equation, and then you work out solutions to the differential equation. And you argue that you know certain kinds of parameters will give you polynomials, and then you get the polynomials and you work out the properties. But in this lecture, we will see how we can get the differential equation corresponding to Legendre polynomials using the standard prescription we have given out ok.

(Refer Slide Time: 01:13)



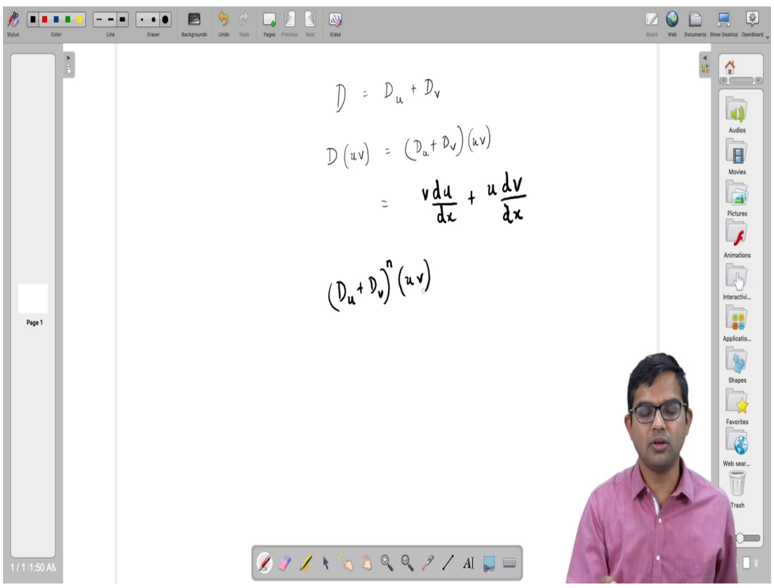
So, the idea is to look for this you know this quantity – work out this quantity, and see and we have already argued that this must be a factor of you know the polynomial equation. So, you must construct this product of these three quantities for Legendre polynomials w of x is equal to 1, s of x is equal to x squared minus 1. So, if you put x squared minus 1 and then multiply by the derivative of this Legendre polynomial of nth order P n of x, and then take the derivative of this object. So, this whole quantity better be directly proportional to the polynomial itself.

And so basically we have to work out this, this constant lambda. This coefficient needs to be determined and that we will do with the use of Rodrigues' formula and a rule which goes by the name of Leibniz rule which is basically an application of the binomial theorem.

So, let us see what the Leibniz rule is. So, the Leibniz rule is useful if you want to take the nth derivative of a product of two functions. We know that the derivative of the product of two functions is given by the product of the derivative of one function and leaving the other as a constant plus the product of the derivative of the other function leaving the first one as a constant right.

So, if you want to take the nth derivative of a product, it turns out that there is this binomial-like formula available. So, you have summation over r going from 0 to n choose r, the rth derivative of the first function u times the n minus rth derivative of the second function v with respect to x right. So, this is in fact a direct consequence of the binomial theorem.

(Refer Slide Time: 03:07)

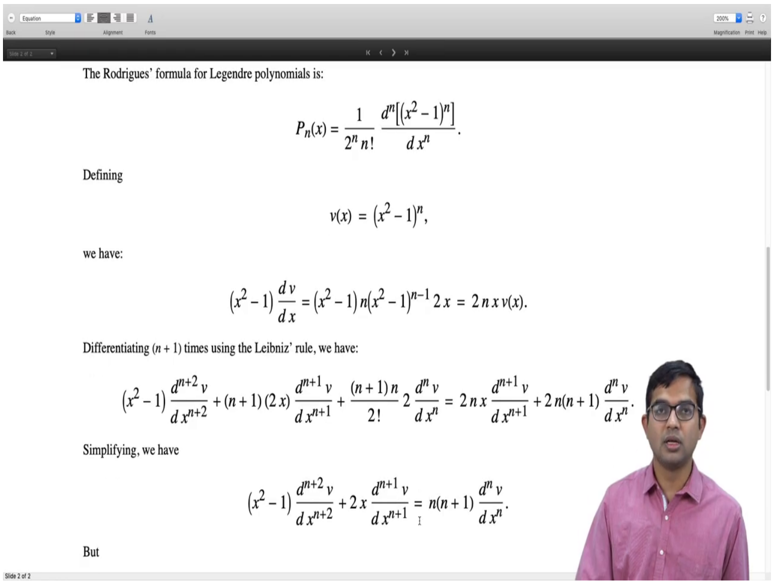


And so one way to see this is to write this you know derivative D that we want to carry as you know D u plus D v right. So, in other words, so if you want to operate this on some function u v we know that this is going to be basically D u plus D v acting on u v. What basic, what this notation really means is you treat u, so when D u acts on u v, it will treat v as a constant. So, you write it is basically like saying v times dou v by dou x. Well, I mean we do not even need, we do not even need let us go back to this.

So, we write it as v times du by dx plus u times dv by dx right. So, we do not even need partial derivatives. So, basically you can introduce this notation and good notation automatically leads you to this result. So, basically what we are interested in finding is D u plus D v the whole power n acting on some product of two functions.

And then you use the binomial theorem to expand it out, and you can verify that indeed the binomial theorem which is being applied to the sum of two operators will give you this relation right. So, you can check this by you know considering increasing powers of n for example, and see for yourself that indeed it holds out. Basically, the idea is quite simple and we will use this rule right.

(Refer Slide Time: 05:13)



So, we start with the Rodrigues' formula for Legendre polynomials which is just 1 over 2 to the n, n factorial times the nth derivative of x squared minus 1 the whole power n. And now we define this function v of x to be x squared minus 1 the whole power n. Now, suppose we take the derivative of this function dv by dx is n times x squared minus 1 the whole power n minus 1.

So, now, it is convenient to multiply this derivative by x squared minus 1, so that you basically get back this v itself right. So, you have x squared minus 1 times dv by dx will be just there is going to be this extra factor 2 times n times x times v of x right. So, this n times x squared minus 1 to the n n minus 1, then you have also 2 x you collect all that and write it as to n x. But then since you have also multiplied by x squared minus 1 you have filled back the lost power. And so you get back v of x right.

So, now we can differentiate this you know this equality n plus 1 times right, so that using the Leibniz's rule. So, now, if we do this the left hand side is you know you think of this is a product of two functions x squared minus 1 is u your u and dv by dx is like your v in this formula that we wrote down.

And so we might think that if you are going to take a derivative with n plus 1 times, we might actually have terms ranging from r equal to 0 all the way up to n plus 1. So, n plus two terms we might think. But it turns out that in this case there are only three terms which survive.

The reason is because we have a quadratic function here. One of these terms is the quadratic function whose 0th derivative is x squared minus 1, then there is a first derivative which will be 2 x, and then the second derivative is 2, but any higher order derivatives are going to just give you 0. So, all those extra terms will cancel.

So, the first term is when you do not take a derivative of this at all that is going to be x squared minus 1 times the n plus 1th derivative of dv by dx which is nothing but the n plus 2th derivative of v plus then we have this coefficient the binomial coefficient which is n plus 1 choose 1 which is just n plus 1, then 2 x from this first term, then n plus 1th, so the nth derivative of dv by dx which is an n plus 1th derivative of v.

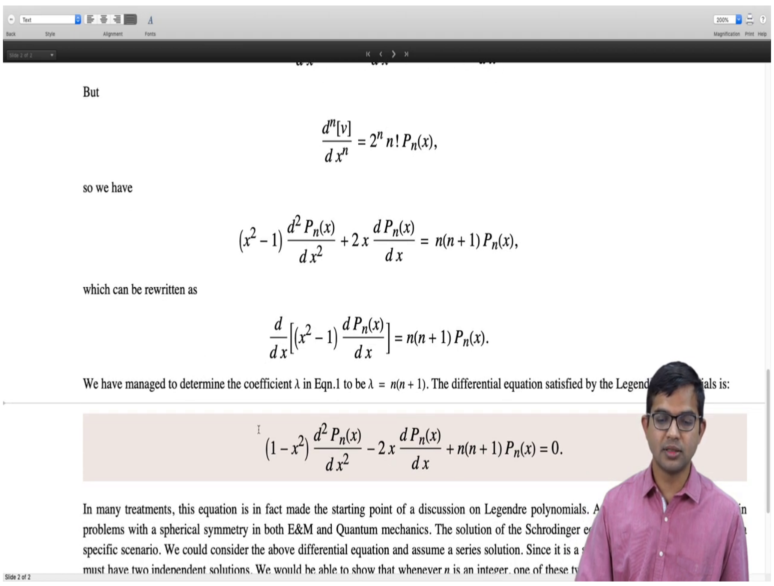
Then we have a the next term is going to be n plus 1 choose 2 which is n plus 1 into n divided by 2 times 2 comes from the second derivative of this function, and then you must take the n minus 1th derivative of this function which is just the nth derivative of v. So, the left hand side is done. So, the n plus 1 derivative of the left hand side gives us three terms. And the n plus 1 derivative of the right hand side gives us only two terms. So, because we have just a linear function, just x is one of them, and the other one is v.

So, the first term is when you do not take a derivative with respect to x, but all the derivatives is entirely borne by v, so that gives us 2 n times x times n plus 1th derivative of v, then we have plus 2 n times n plus 1 n plus 1 choose one. So, the derivative of x is just 1, so then you have to take the nth derivative of v.

So, there are just these two terms on the right hand side. So, both on the left hand side and the right hand side the result has been obtained by applying the Leibniz rule. So, you should check this and convince yourself that it indeed is correct. So, now we simplify this. We see that there is this term for x n plus 2th derivative, then n plus 1th derivative here we have n plus 1 times 2 x, and then we also have 2 n x.

So, you know they can be simplified to just give you 2 x times n plus 1th derivative of v with respect to x. Then this must be equal to you know this stuff can go to the right hand side. So, 2 n into n plus 1 minus so these two have also been cancelled here, so minus n into n plus 1. So, it is just n times n plus 1 times the nth derivative of v on the right hand side.

(Refer Slide Time: 09:38)



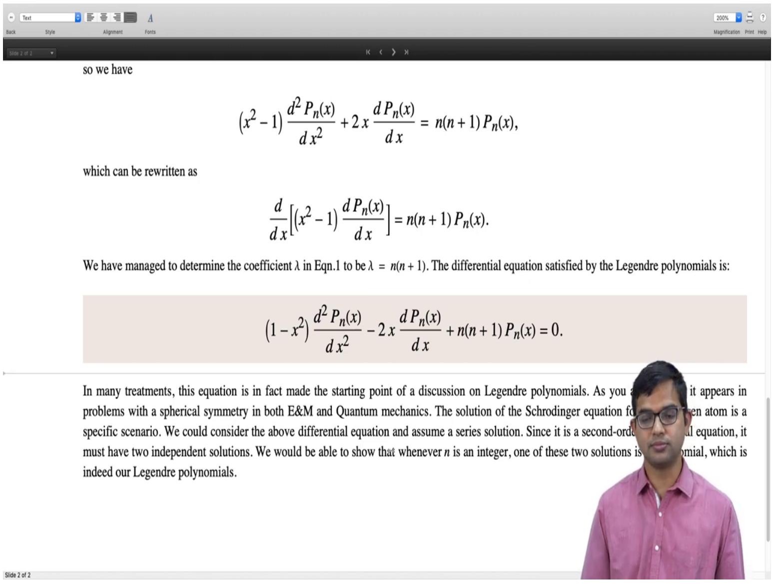
So, now I mean we recall that what is v. v of x is after all x squared minus 1 the whole power n which is really I mean so if you take the nth derivative of v, it is really nothing but the Legendre polynomial itself except that there is this extra factor. So, let us rewrite the nth derivative of v as 2 to the n times n factorial times P n of x and then plug this back in.

So, we see that you know then you have to take one extra derivative here, and two extra derivatives here. And so this factor remains unchanged when you take this derivative of v or second derivative of v. So, all of that will just get cancelled on both sides. And so effectively you are just left with this equation x squared minus 1 times d squared P n of x divided by d x squared plus 2 x times the first derivative of P n of x with respect to x is equal to n times n plus 1 times P n of x.

So, now this can of course, be rewritten as d by dx of x squared minus 1 times d P n of x divided by dx equal to the right hand side as it is. So, a moment's thought here reveals that this is really the same form that we already started with, we wanted to work out this. And so in fact, we have obtained our lambda. So, the lambda here corresponding to the nth, nth polynomial is actually just n times n plus 1 right.

So, this is you can think of this as like an eigenvalue equation. So, there is this operation you know you operate with this stuff on P n we will get back some eigenvalue times the same eigenfunction right. And so only if you choose your lambda to be of this form n into n plus 1 where n is an integer only then do you get polynomials, and those polynomials are the Legendre polynomials.

(Refer Slide Time: 11:33)



So, the differential equation satisfying the Legendre polynomials is exactly this where it is important that this n into n plus 1 both you know appears, and where n is an integer right. So, this is you know we have worked out this differential equation which perhaps we are all familiar with because whenever there is spherical symmetry, and as this is the case in a lot of problems in E M and Quantum Mechanics, you know this differential equation does appear.

And so there you know you start with this differential equation where this constraint is not necessarily already put in. And then we argue that if you want solutions which are polynomial, so that the wave function has a you know physically viable meaning associated with it, then we are interested in getting these Legendre polynomials which turn out to be Legendre polynomials.

And then often that is the starting point from which we make a thorough study of these polynomials and their properties, ok. So, in this lecture we have started with the Legendre polynomials, use some of their properties, and use this general prescription to work out the differential equation corresponding to Legendre polynomials.

Thank you.